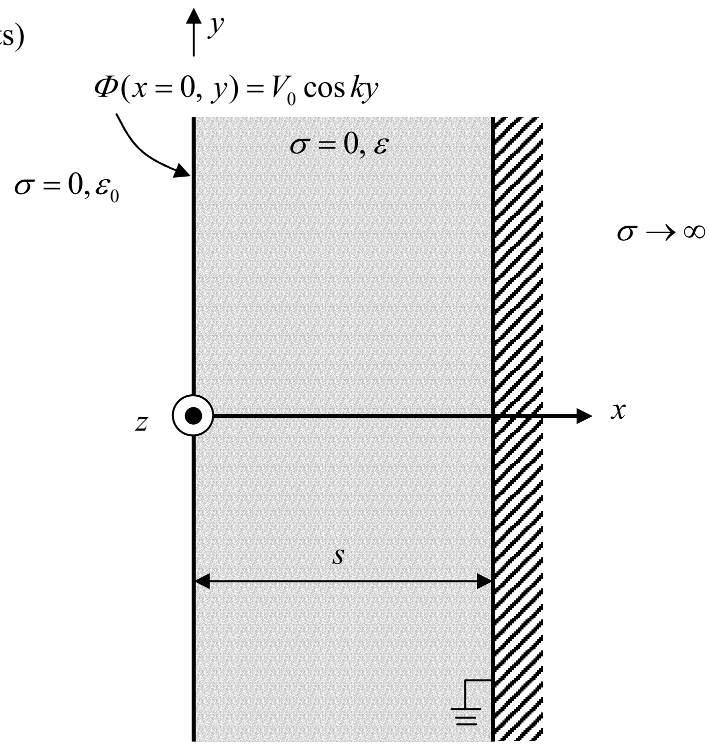
Massachusetts Institute of Technology
Department of Electrical Engineering and Computer Science
6.641 Electromagnetic Fields, Forces, and Motion

Final Exam   May, 17 2004
Spring 2004, 1:30-4:30PM

6.641 Formula Sheet Attached in the study materials section. You are also allowed to use the formula sheets that you prepared for Quiz 1, Quiz 2, and an additional 8 ½"x 11" formula sheet (both sides) that you have prepared for the Final.

Problem 1 (25 points)



A potential sheet of infinite extent in the y and z directions is placed at $x = 0$ and has potential distribution $\Phi(x=0, y) = V_0 \cos ky$. Free space with no conductivity ($\sigma = 0$) and permittivity ε_0 is present for $x < 0$ while for $0 < x < s$ a perfectly insulating dielectric ($\sigma = 0$) with permittivity ε is present. The region for $x > s$ is a grounded perfect conductor at zero potential.

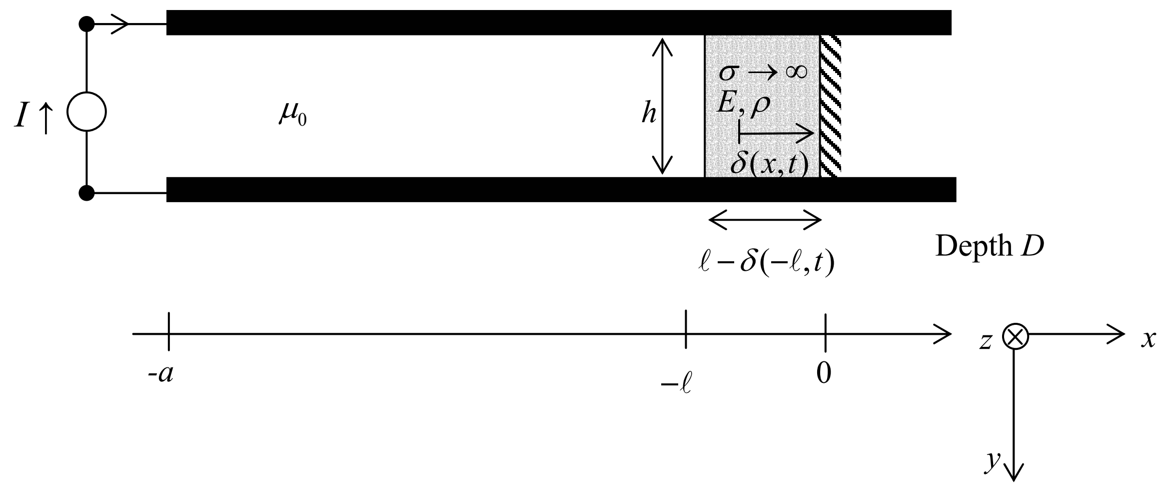
a) What are the potential distributions for $x < 0$ and $0 < x < s$?

b) What are the surface charge densities at $x = 0$, $\sigma_f(x=0, y)$, and at $x = s$, $\sigma_f(x=s, y)$?

c) What is the force, magnitude and direction, on a section of the perfect conductor at $x = s$ that extends over the region $0 < y < \frac{\pi}{k}$ and $0 < z < D$?

Hint: $\int \cos^2 y\,dy = y/2 + (\sin 2y)/4$

Problem 2 (25 points)



Parallel plate electrodes with spacing h and depth D are excited by a DC current source I. An elastic rod surrounded by free space has mass density ρ, modulus of elasticity E, equilibrium length ℓ when $I = 0$, and has infinite ohmic conductivity σ. The elastic rod end at $x = 0$ is fixed while the deflections of the rod are described as $\delta(x,t)$ and are assumed small $|\delta(x,t)| \ll \ell$. The rod width $\ell - \delta(-\ell,t)$ changes as I is changed because of the magnetic force. The DC current flows as a surface current on the $x = -(\ell - \delta(-\ell,t))$ end of the perfectly conducting rod.

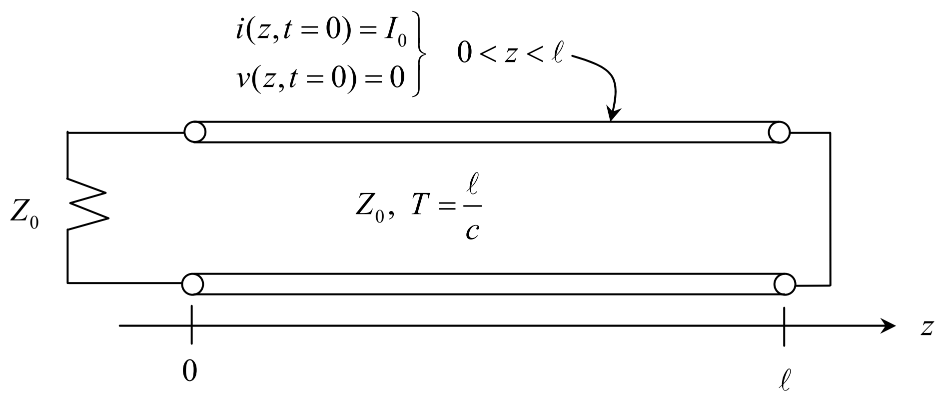
a) Calculate H_z in the free space region $-a < x < -(\ell - \delta(-\ell,t))$. Neglect fringing field effects and assume $h \ll a$ and $h \ll D$.

b) Using the Maxwell Stress Tensor calculate the magnetic force per unit area on the $x = -(\ell - \delta(-\ell,t))$ end of the rod.

c) Calculate the steady state change in rod length $\delta(x = -\ell)$.

d) Noise creates fluctuations $\delta'(x,t)$ in longitudinal displacement. What are the natural frequencies of the rod?

Problem 3 (25 points)



An electrical transmission line of length ℓ has characteristic impedance Z_0. Electromagnetic waves can travel on the line at speed c, so that the time to travel one-way over the line length ℓ is $T=\ell/c$. The line is matched at $z = 0$ and is short circuited at $z=\ell$. At time $t = 0$, a lightning bolt strikes the entire line so that there is a uniform current along the line but with zero voltage:

$$\left.\begin{array}{l} i(z,t=0)=I_0 \\ v(z,t=0)=0 \end{array}\right\} \quad 0<z<\ell$$

Since the voltage and current obey the telegrapher's relations:

$$\frac{\partial v}{\partial z}=-L\frac{\partial i}{\partial t}, \quad c=\frac{1}{\sqrt{LC}}$$

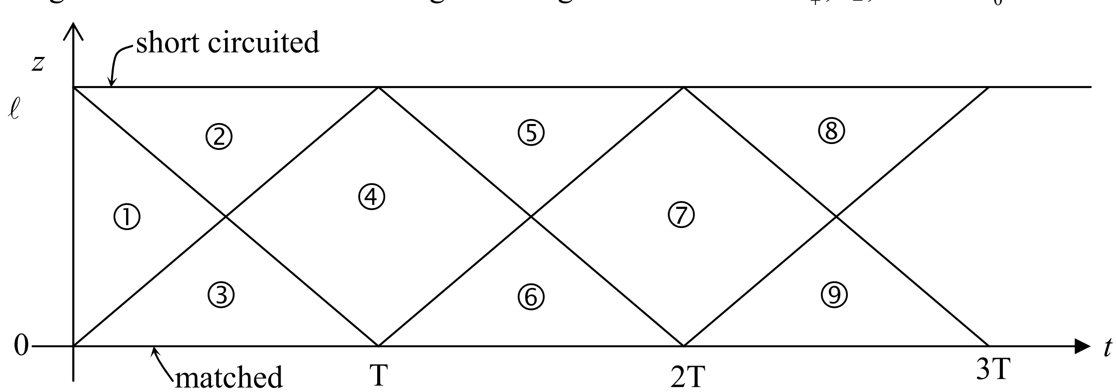
$$\frac{\partial i}{\partial z}=-C\frac{\partial v}{\partial t}, \quad Z_0=\sqrt{L/C}$$

the voltage and current along the line are related as

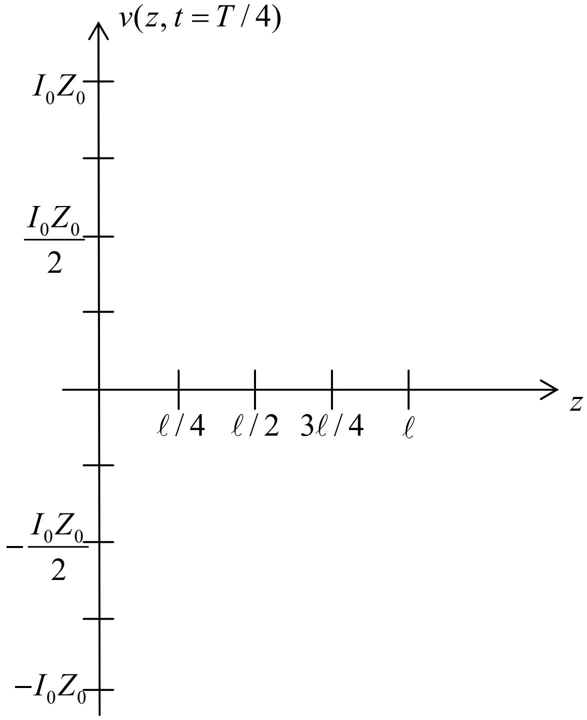
$$v+iZ_0=c_+ \text{ on } \frac{dz}{dt}=c$$

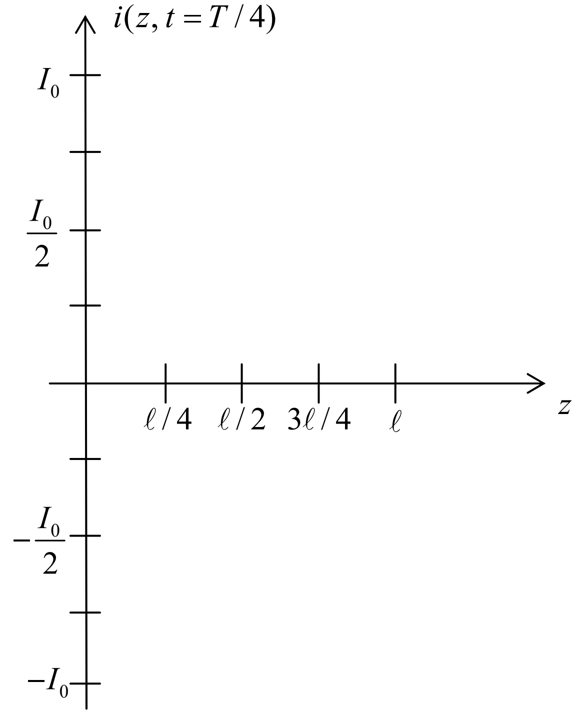
$$v-iZ_0=c_- \text{ on } \frac{dz}{dt}=-c$$

a) The solutions for $v(z,t)$ and $i(z,t)$ can be found using the method of characteristics within each region shown below. Within regions 1-9 give the values of c_+, c_-, v and iZ_0.



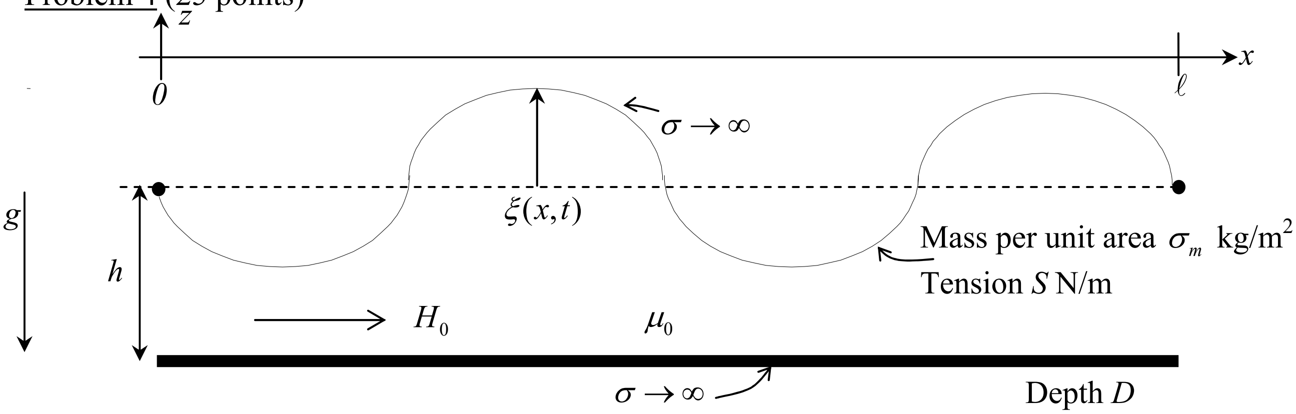
b) Plot $v(z, t = T/4)$ and $i(z, t = T/4)$.





c) How long a time does it take for the transmission line to have $v(z,t) = 0$ and $i(z,t) = 0$ everywhere for $0 < z < \ell$ for all further time?

Problem 4 (25 points)



A perfectly conducting membrane of depth D with mass per unit area σ_m and tension S is a distance h above a rigid perfect conductor. The membrane and rigid conductor are in free space and support currents such that when the membrane is flat, $\xi(x,t)=0$, the static uniform magnetic field intensity is H_0. As the membrane deforms, the flux through the region between membrane and rigid conductor is conserved. The system is in a downward gravity field with gravitational acceleration $\bar{g} = -g\bar{i}_z$. The membrane deflection has no dependence on y and is fixed at its two ends at $x = 0$ and $x = \ell$.

a) Assuming that $\xi(x,t) \ll h$ and that the only significant magnetic field component is x directed, how is $H_x(x,t)$ approximately related to $\xi(x,t)$ to linear terms in $\xi(x,t)$?

b) Using the Maxwell Stress tensor and the result of part (a), to linear terms in small displacement $\xi(x,t)$, what is the z directed magnetic force per unit area, F_z, on the membrane?

c) To linear terms in small displacement $\xi(x,t)$, express the membrane equation of motion in the form

$$a\frac{\partial^2 \xi}{\partial t^2} = b\frac{\partial^2 \xi}{\partial x^2} + c\xi + d$$

What are a, b, c, and d?

d) What value of H_0 is needed so that in static equilibrium, the membrane has no sag, $\xi(x,t)=0$.

Continue to next page for parts (e)-(g)

Prob. 4 continued.

e) About the equilibrium of part (d), what is the $\omega - k$ dispersion relation for membrane deflections of the form

$$\xi(x,t) = \mathrm{Re}[\hat{\xi} e^{j(\omega t - kx)}]?$$

Solve for *k* as a function of ω and system parameters.

f) Using all the values of *k* found in part (e), find a superposition of solutions of the form of $\xi(x,t)$ given in (e), that satisfy the zero deflection boundary conditions at the ends of the membrane at $x = 0$ and $x = \ell$. What are the allowed values for *k*?

g) Is this system always stable or under what conditions can it be unstable? When stable, what are the natural frequencies and if unstable what are the growth rates of the instability?