6.641 Electromagnetic Fields, Forces and Motion
Quiz 1 Solutions
March 12, 2003

1.

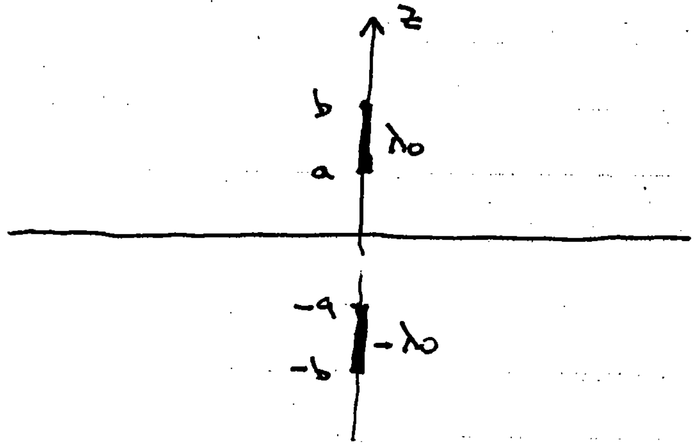


Image line charge $-\lambda_0$ extends from $-a < z < b$. Take charge element $\pm\lambda_0 dz'$ at $z = z'$. For $z > b$, $z > z'$.

a) $$d\Phi(r=0,z) = \frac{\lambda_0}{4\pi\epsilon_0}\left[\frac{1}{z-z'} - \frac{1}{z+z'}\right]dz'$$

$$\Phi(r=0,z) = \frac{\lambda_0}{4\pi\epsilon_0}\int_a^b \left[\frac{1}{z-z'} - \frac{1}{z+z'}\right]dz'$$

$$= \frac{\lambda_0}{4\pi\epsilon_0}\left[-\ln(z'-z) - \ln(z'+z)\right]\Big|_{z'=a}^{b}$$

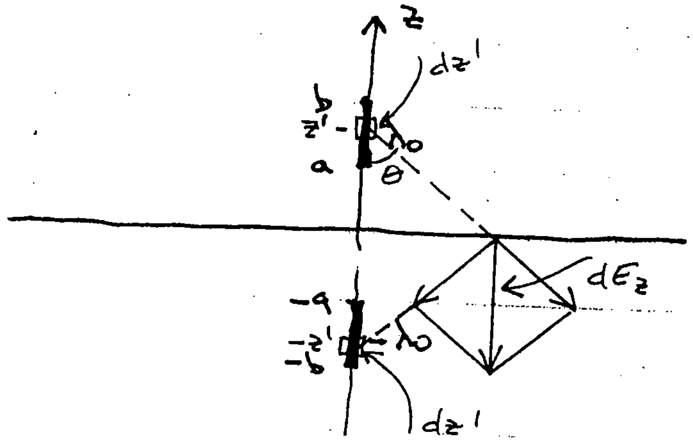
$$= \frac{\lambda_0}{4\pi\epsilon_0}\left[-\ln(b-z) - \ln(b+z) + \ln(a-z) + \ln(a+z)\right]$$

$$= \frac{\lambda_0}{4\pi\epsilon_0}\ln\left[\frac{(a-z)(a+z)}{(b-z)(b+z)}\right] = \frac{\lambda_0}{4\pi\epsilon_0}\ln\left[\frac{a^2-z^2}{b^2-z^2}\right]$$

b) $$E_z(r=0,z) = -\frac{\partial\Phi(r=0,z)}{\partial z} = \frac{\lambda_0}{4\pi\epsilon_0}\left[-\frac{1}{b-z} + \frac{1}{b+z} + \frac{1}{a-z} - \frac{1}{a+z}\right]$$

$$= \frac{\lambda_0 z}{2\pi\epsilon_0}\frac{(b^2-a^2)}{(a^2-z^2)(b^2-z^2)}$$

c) $$\overset{z=0}{dE_z} = \frac{-\lambda_0 dz'}{4\pi\epsilon_0[z'^2+r^2]}\, 2\cos\theta \quad ; \quad \cos\theta = \frac{z'}{\sqrt{z'^2+r^2}}$$



$z>0$

$$dE_z = \frac{-\lambda_0 z' dz'}{2\pi\epsilon_0 [z'^2 + r^2]^{3/2}}$$

$$E_z = \frac{-\lambda_0}{2\pi\epsilon_0} \int_a^b \frac{z' dz'}{[z'^2+r^2]^{3/2}}$$

$$= \frac{\lambda_0}{2\pi\epsilon_0} \left. \frac{1}{\sqrt{z'^2+r^2}} \right|_{z'=a}^{b}$$

$$= \frac{\lambda_0}{2\pi\epsilon_0} \left[\frac{1}{\sqrt{b^2+r^2}} - \frac{1}{\sqrt{a^2+r^2}} \right]$$

$$\sigma_s(z=0) = \epsilon_0 E_z(z=0_+) = \frac{\lambda_0}{2\pi} \left[\frac{1}{\sqrt{b^2+r^2}} - \frac{1}{\sqrt{a^2+r^2}} \right]$$

2. a) $\rho_f(t) = \rho_0 e^{-t/\tau}$, $\tau = \epsilon/\sigma$

b) $\nabla \cdot \overline{E} = \frac{\partial E_x}{\partial x} \overset{!}{=} \frac{\rho_f(t)}{\epsilon} \Rightarrow E_x = \frac{\rho_f(t) x}{\epsilon} + C(t)$

$$\int_0^s E_x dx = \frac{\rho_f(t) s^2}{2\epsilon} + C(t) s = 0$$

$$C(t) = -\frac{\rho_f(t) s}{2\epsilon}$$

$$E_x = \frac{\rho_f(t)}{\epsilon}\left(x - \frac{s}{2}\right)$$

c) $\sigma_s(x=0) = \epsilon E_x(x=0) = -\frac{\rho_f(t) s}{2}$

$\sigma_s(x=s) = -\epsilon E_x(x=s) = -\frac{\rho_f(t) s}{2}$

d) $$\frac{i(t)}{A} = \sigma E_x(x=s) + \epsilon \left.\frac{\partial E_x}{\partial t}\right|_{x=s}$$

$$= \sigma \frac{\rho_f(t) s}{2\epsilon} + \frac{\cancel{\epsilon} s}{2 \cancel{\epsilon}} \frac{\partial \rho_f}{\partial t}$$

$$= \frac{\sigma s}{2\epsilon} \rho_0 e^{-t/\tau} + \frac{s}{2}\left(-\frac{1}{\tau}\right) \rho_0 e^{-t/\tau}$$

$$= \frac{\sigma s}{2\epsilon} \rho_0 e^{-t/\tau} - \frac{s\sigma}{2\epsilon} \rho_0 e^{-t/\tau}$$

$$= 0$$

Problem 3

(A) $\bar{J}$ is only in the $\hat{y}$ direction so therefore $\bar{A}$ is only in the $\hat{y}$ direction. The system is symmetric in y, so only x and z dependencies are expected. Therefore, $\bar{A} = A_y(x,y)\hat{y}$. Following $\bar{B} = \mu_0 \bar{H} = \nabla \times \bar{A}$,
$\bar{B} = \hat{z}\frac{\partial A_y}{\partial x} - \hat{x}\frac{\partial A_y}{\partial z}$. In summary,
$\bar{B} = B_x(x,z)\,\hat{x} + B_z(x,z)\,\hat{z}$.

(B)
$$\bar{H}(\bar{r}) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\bar{K}(\bar{r}') \times (\bar{r}-\bar{r}')}{4\pi\,|\bar{r}-\bar{r}'|^3}\,dx'\,dy'$$

$$\bar{K}(\bar{r}') = \begin{cases} K_0\,\hat{y} & \text{for } x' > 0 \\ -K_0\,\hat{y} & \text{for } x' < 0 \end{cases}$$

$\bar{r} = x\hat{x} + z\hat{z}$ (y dependence is not necessary)
$\bar{r}' = x'\hat{x} + y'\hat{y}$ (integration only in the x-y plane)

$$\bar{H}(\bar{r}) = \int_0^{\infty}\int_{-\infty}^{\infty} \frac{K_0\hat{y} \times ((x-x')\hat{x} - y'\hat{y} + z\hat{z})}{4\pi\left[(x-x')^2 + (y')^2 + (z)^2\right]^{3/2}}\,dy'\,dx'$$
$$- \int_{-\infty}^{0}\int_{-\infty}^{\infty} \frac{K_0\hat{y} \times ((x-x')\hat{x} - y'\hat{y} + z\hat{z})}{4\pi\left[(x-x')^2 + (y')^2 + (z)^2\right]^{3/2}}\,dy'\,dx'$$

$$\bar{H}(x,z) = \int_0^{\infty}\int_{-\infty}^{\infty} \frac{K_0 z\hat{x} - K_0(x-x')\hat{z}}{2\pi\left[(x-x')^2 + (y')^2 + (z)^2\right]^{3/2}}\,dy'\,dx'$$

(C) Let K^* be the distribution of the surface current in the $\hat{y}$ direction at $z=0$ in Parts (A) and (B). Then, the distribution of the surface current in the $\hat{y}$ direction at $z=D$ in this part is given by $K = \frac{1}{2}(K^*(x-L, y) + K^*(x+L, y))$. Here, superposition is used to create the new surface current distribution. The resulting magnetic fields are similarly superimposed, after translation by D in the $\hat{z}$ direction. A second set of superpositions is used to match the boundary condition at the perfect conductor following the method of images. Hence,

$$\bar{H} = \frac{1}{2}\Big[\underbrace{\bar{H}^*(x-L, z-D) + \bar{H}^*(x+L, z-D)}_{\substack{\text{Translated and superimposed}\\ \text{original solutions}}} \underbrace{- \bar{H}^*(x-L, z+D) - \bar{H}^*(x+L, z+D)}_{\text{Image solutions}}\Big]$$

where $\bar{H}^*$ is the magnetic field found in Part B.