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6.641 Electromagnetic Fields, Forces, and Motion
Spring 2005

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Problem Set 9 - Questions

Problem 9.1 (W&M Prob 7.9)

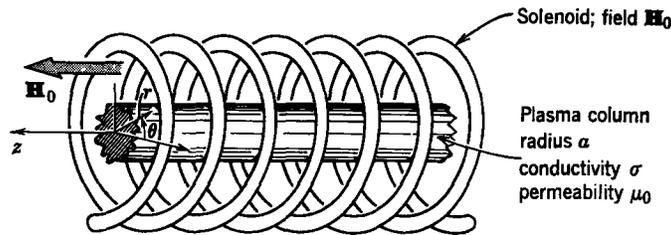


Fig. 7P.9a

Figure 1: A cylindrical column of plasma within a long solenoid

As a possible plasma containment scheme, it has been proposed to create a cylindrical column of plasma within a long solenoid. The plasma is created by use of an arc discharge through a gas. After the plasma is formed the solenoid is excited by rapid discharge of a capacitor bank connected to the terminals. The magnetic field of the solenoid is initially excluded from the interior of the plasma column by surface currents which flow in the theta direction on the plasma; for this reason the device is called a “Theta-pinch machine.” Because the plasma has only a finite conductivity, the magnetic field diffuses into the interior. Therefore to design the machine it is necessary to determine the time associated with this field diffusion. In Fig. 7P.9a the plasma column is shown in the magnetic field. Neglect end effects by assuming an infinitely long system. Furthermore, assume that the plasma remains stationary during the diffusion process.

A

Write an equation governing the magnetic flux density \mathbf{B} inside the plasma.

B

Because there are only θ currents independent of θ , assume that $\mathbf{B} = \hat{\mathbf{z}}B(r, t)$, where $B(r, t)$ has the form

$$B(r, t) = \mu_0 H_0 - \hat{B}(r)e^{-\alpha t}, \quad \alpha > 0$$

which satisfies the condition that $B(r, t) \rightarrow \mu_0 H_0$ as $t \rightarrow \infty$. Using the results of part (a), write an equation for $\hat{B}(r)$.

C

The equation obtained in part (b) for $\hat{B}(r)$ is called “Bessel’s equation of zeroth order.” The general solution to this equation is

$$\hat{B}(r) = C_1 J_0(\sqrt{\mu_0 \sigma \alpha} r) + C_2 N_0(\sqrt{\mu_0 \sigma \alpha} r),$$

Table 1: Roots of $J_0(v_i) = 0$ and the Corresponding Values of $J_1(v_i)$

i	v_i	$J_1(v_i)$
1	2.4048	0.5191
2	5.5201	-0.3403
3	8.6537	.2715
4	11.7915	-0.2325
5	14.9309	0.2065
6	18.0711	-0.1877

where J_0 is the Bessel function of zeroth order and the first kind and N_0 is the Bessel function of zeroth order and the second kind (sometimes called a “Neumann function”). See Fig. 7P.9b. Using the boundary condition at $r = 0$, argue that $C_2 = 0$, hence that $\hat{B}(r) = C_1 J_0(\sqrt{\mu_0 \sigma \alpha} r)$.

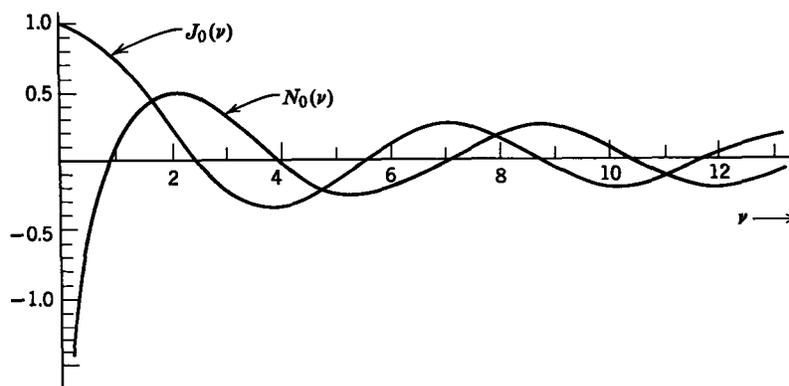


Fig. 7P.9b

Figure 2: Bessel functions of zeroth order

D

Now apply the boundary condition at $r = a$ that $B(r, t) = \mu_0 H_0$ for all $t > 0$ and show that $J_0(\sqrt{\mu_0 \sigma \alpha} a) = 0$. This transcendental equation determines the allowed values of α . The values of α may be obtained from Table 1 as

$$\sqrt{\mu_0 \sigma \alpha} a = v_i; \quad \alpha = \frac{1}{\mu_0 \sigma} \left(\frac{v_i}{a} \right)^2$$

where $J_0(v_i) = 0$.

E

On examination of the results, it is evident that $\hat{B}(r) = C_i J_0 \left[v_i \left(\frac{r}{a} \right) \right]$ is a solution for any of the allowed values of v_i . The most general solution for $B(r, t)$ is obtained by using superposition; hence

$$B(r, t) = \mu_0 H_0 - \sum_{i=1}^{\infty} C_i J_0 \left(v_i \frac{r}{a} \right) \exp \left(- \frac{v_i^2 t}{\mu_0 \sigma a^2} \right)$$

and to evaluate the constants C_i the last boundary condition $B(r, t = 0) \equiv 0$ is used. This condition implies that

$$\sum_{i=1}^{\infty} C_i J_0 \left(v_i \frac{r}{a} \right) = \mu_0 H_0$$

which is just a series in terms of the functions $J_0 \left[v_i \left(\frac{r}{a} \right) \right]$. Using the integrals

$$\int_0^a r J_0 \left(v_i \frac{r}{a} \right) J_0 \left(v_j \frac{r}{a} \right) dr = \begin{cases} \frac{a^2}{2} J_1^2(v_i) & v_i = v_j \\ 0 & v_i \neq v_j \end{cases}$$

and

$$\int_0^a r J_0 \left(v_i \frac{r}{a} \right) dr = \frac{a^2}{v_i} J_1(v_i)$$

evaluate C_1, C_2 and C_3 .

F

What is the fundamental time constant for the diffusion? Evaluate this time constant for $a = 5$ cm and $\sigma = 10^4 / 4\pi (\Omega m)^{-1}$.

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 Woodson, Herbert H., and James R. Melcher. *Electromechanical Dynamics, Part 2: Fields, Forces, and Motion*. Malabar, FL: Kreiger Publishing Company, 1968. ISBN: 9780894644597.

Problem 9.2 (W&M Prob 8.7)

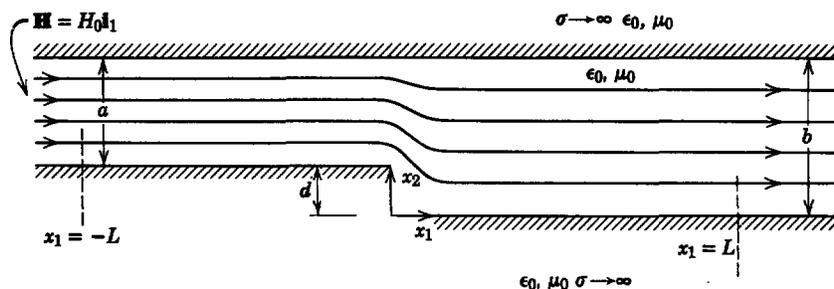


Fig. 8P.7

Figure 3: Two perfectly conducting plates with an applied magnetic field

Two perfectly conducting plates are arranged as shown in Fig. 8P.7. A magnetic field trapped between the plates is established in such a way that it does not penetrate the perfectly conducting plates. Also $H_3 = 0$ and $\frac{\partial}{\partial x_3} = 0$. Under the assumption that $b \ll L$, find the x_1 -component of the force per unit x_3 on the section of the lower plate between $x_1 = L$ and $x_1 = -L$. You may assume that, when $x_1 = -L$, $\mathbf{H} = H_0 i_1$, where H_0 is a known constant.

Courtesy of Herbert Woodson and James Melcher. Used with permission.
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Problem 9.3 (W&M Prob 8.11)

Two parallel conducting plates with a potential difference V_0 are shown in Fig. 8P.11. Assuming that $c < b < a \ll l \ll D$ and that the fringing fields are zero at the extreme points A and B , find the force in the x_1 direction on the lower plate.

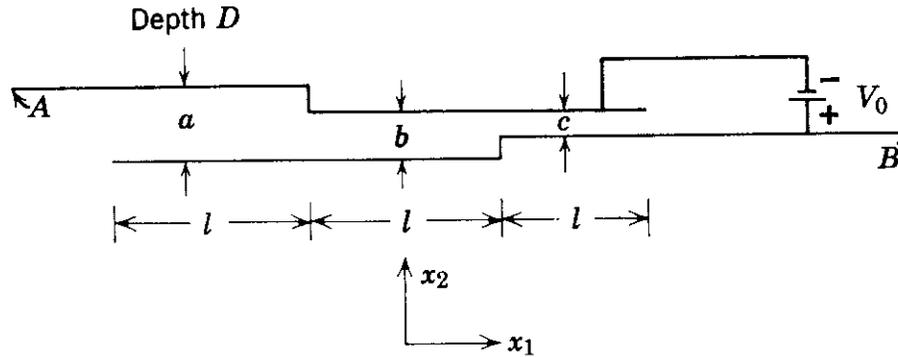


Fig. 8P.11

Figure 4: Two parallel conducting plates with potential difference V_0

Courtesy of Herbert Woodson and James Melcher. Used with permission.

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Problem 9.4 (W&M Prob 8.12)

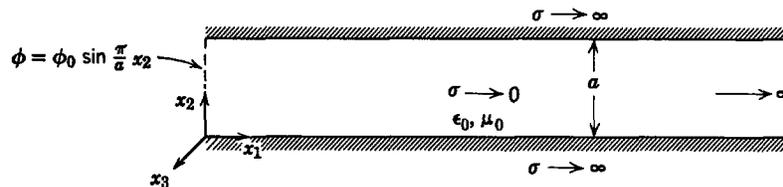


Fig. 8P.12

Figure 5: Parallel perfectly conducting electrodes with imposed potential distribution along $x_1 = 0$

In Fig. 8P.12 two parallel perfectly conducting electrodes extend from $x_1 = 0$ to $x_1 = \infty$ and are infinite in the x_3 direction. The separation of the electrodes in the x_2 direction is a . A potential $\phi = \phi_0 \sin\left(\frac{\pi}{a}\right) x_2$ is established along the x_2 -axis at $x_1 = 0$.

A

Find the electric field intensity \mathbf{E} everywhere between the plates and sketch.

B

Find the total force on the bottom plate per unit depth in the x_3 direction.

C

Find the total force on the top plate per unit depth in the x_3 -direction.

Courtesy of Herbert Woodson and James Melcher. Used with permission.

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Problem 9.5 (W&M Prob 7.14)

You are working on a transportation project and are asked to make an analysis of the following basic method of both levitating and propelling a train. The train rides just above a track which is composed of a slab

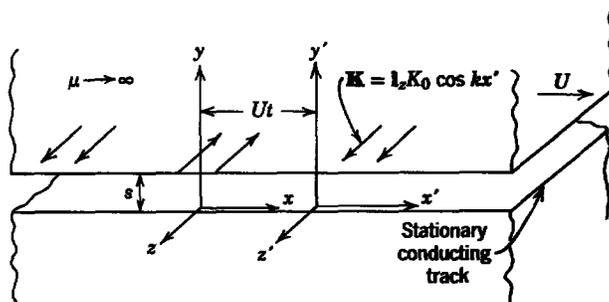


Fig. 7P.14

Figure 6: Magnetic levitation and propulsion

with conductivity σ . Superconducting coils within the train are arranged to produce a current that can be represented by the current sheet \mathbf{K} (see Fig. 7P.14). This current sheet is backed by a highly permeable magnetic shield ($\mu \rightarrow \infty$) which is also attached to the train. (The shield prevents magnetization of the passengers' watches and the attendant possibility that $t \neq t'$.) Here x' is the distance measured with respect to the moving train. Hence, because the train is moving in the x -direction with velocity U , $x = Ut + x'$. We wish to compute the time average force per unit area that presumably holds the train a distance s ($ks \ll 1$) above the track.

A

Express the surface current in the fixed frame $\mathbf{K}(x, t)$.

B

Assume that the track is infinitely thick in the y -direction (under what conditions is this a good assumption?) and compute the magnetic field and current in the conducting track. Assume that $\frac{\partial}{\partial z} = 0$.

C

Compute the time average force per unit area (in the $x - z$ plane) that holds up the vehicle.

D

Compute the force per unit ($x - z$) area that tends to propel the train in the x - direction. Do you see any basic problems with the proposed scheme of propulsion?

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Problem 9.6 (W&M Prob 8.21)

In problem 7.14 a vehicle system was proposed in which a magnetic field provided both suspension (i.e., levitation) and propulsion forces. There it was assumed that the condition $ks \ll 1$ is valid and, to calculate the volume force density, $\mathbf{J} \times \mathbf{B}$ was applied. The Maxwell stress tensor provides an alternate and useful method for the calculation of the forces per unit area (Fig. 8P.21). The solution for the magnetic field in the region $-\infty < y < 0$ is

$$B_x = \text{Re} \left[\mu_0 K_0 e^{\alpha y} e^{jk(x-Ut)} \right]$$

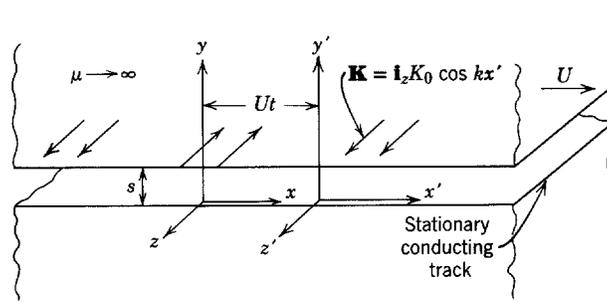


Fig. 8P.21

Figure 7: Proposed vehicle system using magnetic levitation

and

$$B_y = \text{Re} \left[\frac{-jk\mu_0 K_0}{\alpha} e^{\alpha y} e^{jk(x-Ut)} \right]$$

where

$$\alpha = k \left(1 - j \frac{\mu_0 \sigma U}{k} \right)^{\frac{1}{2}}$$

A

Write the components of the Maxwell stress tensor explicitly in terms of B_x and B_y . Present your results in matrix form.

B

Using the stress tensor, compute the time average force per unit area (in the $x - z$ plane) that holds the vehicle up. Take advantage of the periodic variation with x to define a suitable surface.

C

Again using the stress tensor, compute the time average force per unit area ($x - z$) that tends to propel the train.

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Problem 9.7

The figure below shows a magnetohydrodynamic generator. In this generator, a fluid having conductivity σ and free-space permeability is pumped through a rectangular channel with velocity U in the \hat{x} direction. The width and height of the channel are W and D respectively. The channel passes through the gap of a perfectly-permeable C-core of width T in the \hat{x} direction. The C-core is excited by a perfectly-conducting N -turn field coil that carries the current i_F and has a terminal voltage v_F . The two side walls of the channel make perfect electrical contact with the fluid over the width T as the channel passes through the C-core. The current through these armature contacts is i_A and the voltage across them is v_A . A two-dimensional view of the MHD generator is shown in Figure 9.

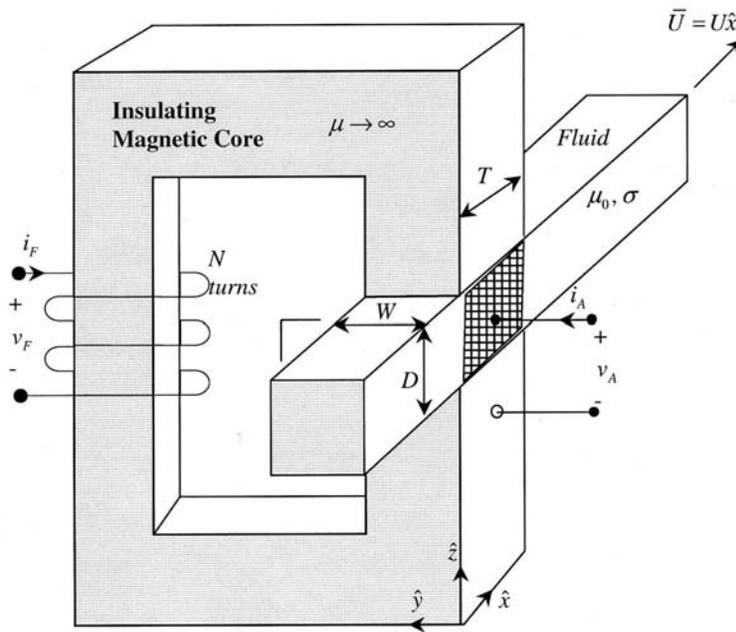


Figure 8: Magnetohydrodynamic generator

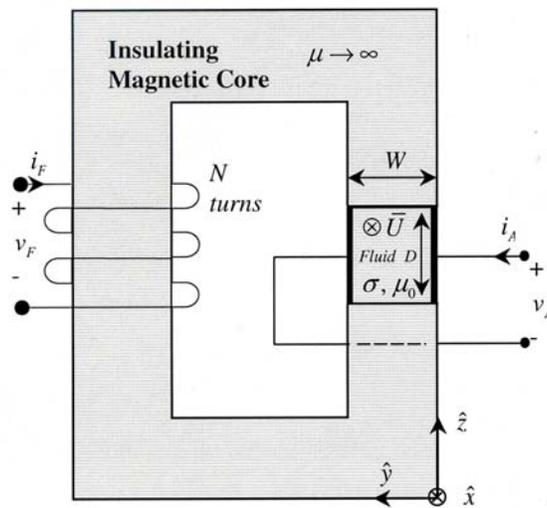


Figure 9: Two-dimensional view of the magnetohydrodynamic generator

A

Determine the \hat{z} -directed magnetic flux density in the gap of the C-core in terms of the field current i_F , and the parameters of the generator. Make reasonable magnetic circuit approximations, and ignore the flux density sourced by the armature current.

B

Determine the self-inductance of the field coil in terms of the parameters of the generator.

C

The static terminal relation for the armature takes the form

$$v_A = Ri_A + GUi_F$$

Determine R and G in terms of the parameters of the generator.

Hint: $\bar{J} = \sigma(\bar{E} + \bar{v} \times \bar{B})$

D

Determine the mechanical power that is required to pump the fluid through the channel in terms of i_A, i_F, U , and the parameters of the MHD generator.

E

The generator is connected such that $i_F = -i_A$ and $v_F = v_A$ as shown below, in an effort to produce self-excitation. For what range of U will it exhibit such self-excitation? Ignore any armature inductance.

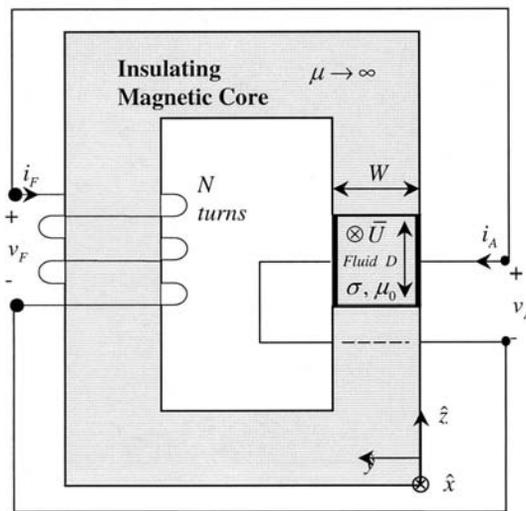


Figure 10: An effort to produce self-excitation