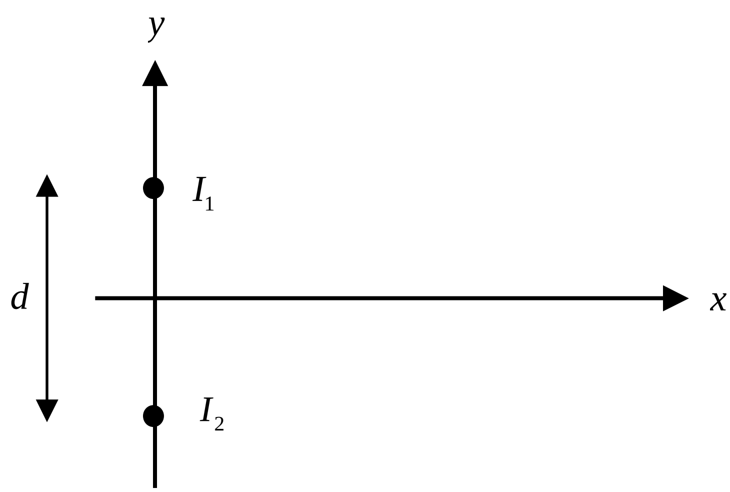
Massachusetts Institute of Technology
Department of Electrical Engineering and Computer Science
6.641 Electromagnetic Fields, Forces, and Motion

Problem Set #2 — Issued: 2/8/05
Spring Term 2005 — Due: 2/17/05

Suggested Reading Assignment: Zahn – 2.3.5, 2.5, 2.6, 2.7, 5.2.4, 5.4

Problem 2.1



(a) Two line currents of infinite extent in the *z* direction are a distance d apart along the *y*-axis. The current I_1 is located at $y=d/2$ and the current I_2 is located at $y=-d/2$. Find the magnetic field (magnitude and direction) at any point in the $y=0$ plane when the currents are:

i) $I_1=I$, $I_2=0$
ii) both equal, $I_1=I_2=I$
iii) of opposite direction but equal magnitude, $I_1=-I_2=I$. This configuration is called a current line dipole with moment $m_x=Id$.

Hint: In cylindrical coordinates $\bar{i}_\phi = \left[-y\bar{i}_x + x\bar{i}_y\right]/\left[x^2+y^2\right]^{\frac{1}{2}}$

(b) For each of the three cases in part (a) find the force per unit length on I_1.

Problem 2.2

The superposition integral for the electric scalar potential is

$$\Phi(\bar{r}) = \int_{V'} \frac{\rho(\bar{r}')dV'}{4\pi\varepsilon_o |\bar{r} - \bar{r}'|} \tag{1}$$

The electric field is related to the potential as

$$\bar{E}(\bar{r}) = -\nabla\Phi(\bar{r}) \tag{2}$$

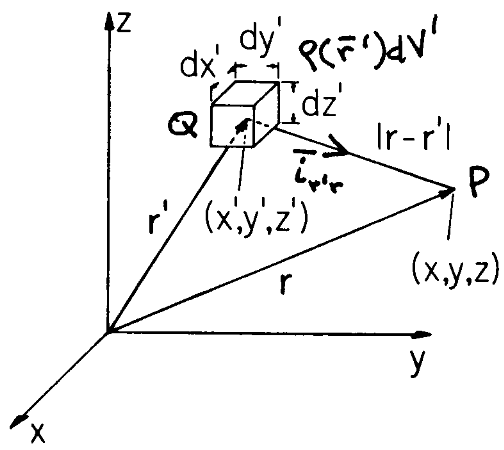


Figure 4.5.1 An elementary volume of charge at **r′** gives rise to a potential at the observer position **r**.

Figure 4.5.1 from *Electromagnetic Fields and Energy* by Hermann A. Haus and James R. Melcher.

The vector distance between a source point at *Q* and a field point at *P* is:

$$\bar{r} - \bar{r}' = (x - x')\bar{i}_x + (y - y')\bar{i}_y + (z - z')\bar{i}_z \tag{3}$$

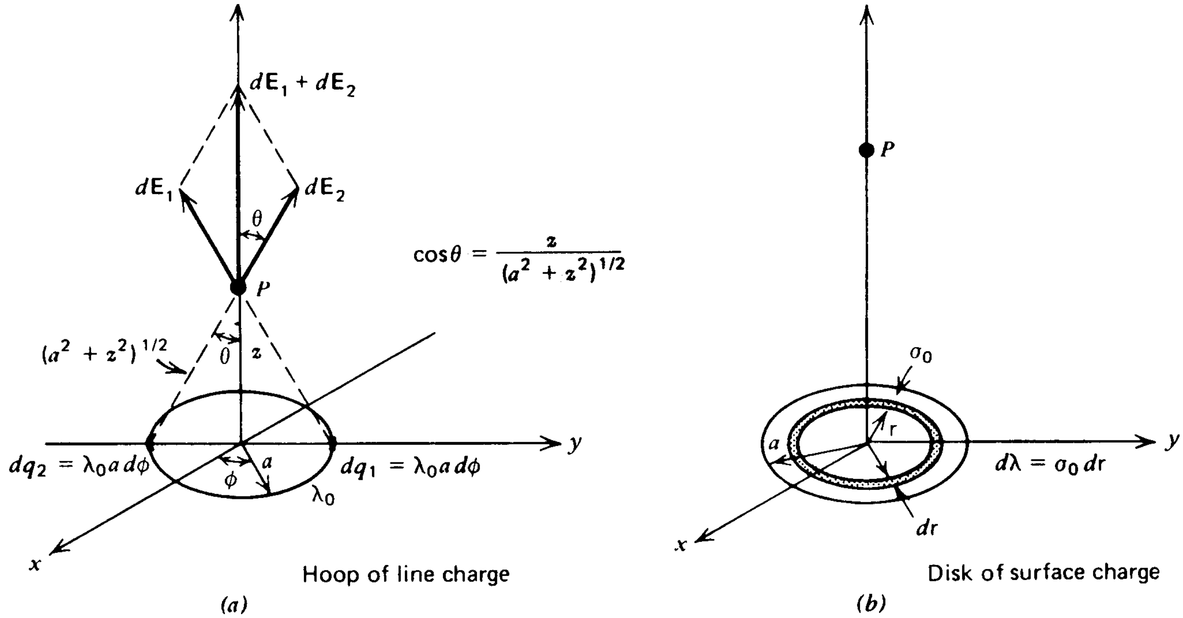
(a) By differentiating $|\vec{r} - \vec{r}'|$ in Cartesian coordinates with respect to the unprimed coordinates at *P* show that

$$\nabla\left(\frac{1}{|\vec{r} - \vec{r}'|}\right) = \frac{-(\bar{r} - \bar{r}')}{|\vec{r} - \vec{r}'|^3} = \frac{-\bar{i}_{r'r}}{|\vec{r} - \vec{r}'|^2} \tag{4}$$

where $\bar{i}_{r'r}$ is the unit vector pointing from *Q* to *P*.

(b) Using the results of (a) show that

$$\bar{E}(\bar{r}) = -\nabla\Phi(\bar{r}) = -\int_{V'} \frac{\rho(\bar{r}')}{4\pi\varepsilon_o}\nabla\left(\frac{1}{|\bar{r} - \bar{r}'|}\right)dV' = \int_{V'} \frac{\rho(\bar{r}')\bar{i}_{r'r}}{4\pi\varepsilon_o |\bar{r} - \bar{r}'|^2}dV' \tag{5}$$



Figures a & b from: *Electromagnetic Field Theory: A Problem Solving Approach*, by Markus Zahn, Robert E. Krieger Publishing Company, 1987. Used with permission.

(c) A circular hoop of line charge λ_0 coulombs/meter with radius a is centered about the origin in the $z=0$ plane. Find the electric scalar potential along the z-axis for $z<0$ and $z>0$ using Eq. (1) with $\rho(r')dV' = \lambda_o a d\phi$. Then find the electric field magnitude and direction using symmetry and $\overline{E} = -\nabla\Phi$. Verify that using Eq. (5) gives the same electric field. What do the electric scalar potential and electric field approach as $z \to \infty$ and how do these results relate to the potential and electric field of a point charge?

(d) Use the results of (c) to find the electric scalar potential and electric field along the z axis for a uniformly surface charged circular disk of radius a with uniform surface charge density σ_0 *coulombs/m*^{2}. Consider $z>0$ and $z<0$.

(e) What do the electric scalar potential and electric field approach as $z \to \infty$ and how do these results relate to the potential and electric field of a point charge?

(f) What do the potential and electric field approach as the disk gets very large so that $a \to \infty$.

Problem 2.3

The curl and divergence operations have a simple relationship that will be used throughout the subject.

(a) One might be tempted to apply the divergence theorem to the surface integral in Stokes' theorem. However, the divergence theorem requires a closed surface while Stokes' theorem is true in general for an open surface. Stokes' theorem for a closed surface requires the contour to shrink to zero giving a zero result for the line integral. Use the divergence theorem applied to the closed surface with vector $\nabla \times \vec{A}$ to prove that $\nabla \bullet (\nabla \times \vec{A}) = 0$.

(b) Verify (a) by direct computation in Cartesian and cylindrical coordinates.

Problem 2.4

Charge is distributed along the z axis such that the charge per unit length $\lambda_l(z)$ is given by

$$\lambda_1(z) = \begin{cases} \dfrac{\lambda_o z}{a} & -a < z < a \\ 0 & z < -a;\ z > a \end{cases}$$

(a) What is the total charge?

(b) Determine the electric scalar potential Φ and electric field $\bar{E}$ along the *z*-axis for $z > a$.

Hint: $\int \frac{z'}{z - z'} dz' = -z' - z \ln(z' - z)$

(c) What do the electric scalar potential and electric field approach as $z \to \infty$ and how do these results relate to part (a)? Note that you have to use the series expansions below up to third order in some cases.

Hints:

$$\ln[1+\delta] = \delta - \frac{1}{2}\delta^2 + \frac{1}{3}\delta^3 + \cdots \quad , \ |\delta| < 1$$

$$\ln\left[\frac{1+\delta}{1-\delta}\right] = 2\left[\delta + \frac{\delta^3}{3} + \cdots\right] \quad , \ |\delta| < 1$$

$$\frac{1}{1-\delta} \approx \left[1 + \delta + \delta^2 + \delta^3 + \cdots\right] \quad , \ |\delta| < 1$$

(d) What is the effective dipole moment of this charge distribution?