6.641 Electromagnetic Fields, Forces, and Motion
Spring 2005

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Problem Set 11 - Solutions	
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Problem 11.1

A

The given equations follow by writing out Maxwell's equations and assuming $\bar{E}$ and $\bar{H}$ have the given directions and dependences.

B

The force equation for an incremental volume element is

$$\bar{F} = \bar{i}_x m n_e \frac{\partial v_x}{\partial t}$$

where $\bar{F}$ is the force density due to electrical forces on the electrons

$$\bar{F} = -\bar{i}_x e n_e E_x$$

Thus,

$$-e n_e E_x = m n_e \frac{\partial v_x}{\partial t} \tag{1}$$

C

As the electrons move, they give rise to the current density

$$J_x \approx -e n_e v_x \qquad \text{(linearized)} \tag{2}$$

D

Assume $e^{j(\omega t - kx)}$ dependence and (1) and (2) require

$$\hat{J}_x = -j\frac{e^2 n_e}{\omega m}\hat{E}_x$$

$$= -j\omega\varepsilon_0 \left[\frac{\omega_p^2}{\omega^2}\right]\hat{E}_x$$

where $\omega_p = \sqrt{\frac{e^2 n_e}{m\varepsilon_0}}$ is called the plasma frequency. (See page 600 in Woodson and Melcher, Electromechanical Dynamics, vol. 2). Combining this with Maxwell's equations:

$$k^2 = \frac{\omega^2}{c^2}\left[1 - \frac{\omega_p^2}{\omega^2}\right]; \qquad c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}}$$

E

We have a dispersion which yields evanescent waves below the plasma (cutoff) frequency. Below this frequency, the electrons respond to the electric field associated with the wave in such a way as to reflect rather than transmit an incident electromagnetic wave.

F

Waves impinging on a boundary between free space and plasma will be totally reflected if the wave frequency $\omega < \omega_p$. The plasma frequency for the ionosphere is typically $f_p \approx 10\,\mathrm{MHz}$. This result explains why AM broadcasts ($500\,\mathrm{kHz} < f < 1500\,\mathrm{kHz}$) can commonly be monitored all over the world, whereas FM ($88\,\mathrm{MHz} < f < 108\,\mathrm{MHz}$) has a range limited to "line-of-sight."

Problem 11.2

A

The equation of motion for the string is

$$m\frac{\partial^2\xi}{\partial t^2} = f\frac{\partial^2\xi}{\partial x^2} + S - mg \tag{3}$$

where, for small deflections ξ in the "$\frac{1}{r}$" field from Q,

$$S \approx \frac{qQ}{2\pi\varepsilon_0 d}\left[1 + \frac{\xi}{d}\right]$$

In static equilibrium, $\xi = 0$ and from (3)

$$qQ = 2\pi d\varepsilon_0 \cdot mg \tag{4}$$

B

The perturbation equation of motion remains

$$m\frac{\partial^2\xi}{\partial t^2} = f\frac{\partial^2\xi}{\partial x^2} + \left(\frac{qQ}{2\pi d^2\varepsilon_0}\right)\xi \tag{5}$$

Assume $e^{j(\omega t - kx)}$ dependence and (5) requires ($v_s = \sqrt{\frac{f}{m}}$)

$$\omega^2 = v_s^2 k^2 - \frac{qQ}{2\pi d^2\varepsilon_0 m}$$

or from (4),

$$\omega^2 = v_s^2 k^2 - \frac{g}{d}$$

The boundary conditions require $k = \frac{n\pi}{l}$, and for stability the mot critical mode is $n = 1$; thus

$$v_s^2\left(\frac{\pi}{l}\right)^2 > \frac{g}{d}$$

$$m < \frac{fd}{g}\left(\frac{\pi}{l}\right)^2$$

C

Increase f, d, or decrease l.

Problem 11.3

A

This problem is very similar to that of problem 10.7. Using the same reasoning as in that problem, we obtain

$$\sigma_m \frac{\partial^2 \xi_1}{\partial t^2} = S\frac{\partial^2 \xi_1}{\partial x^2} + \frac{\varepsilon_0 V_0^2}{d^3}(2\xi_1 - \xi_2)$$

$$\sigma_m \frac{\partial^2 \xi_2}{\partial t^2} = S\frac{\partial^2 \xi_2}{\partial x^2} + \frac{\varepsilon_0 V_0^2}{d^3}(2\xi_2 - \xi_1)$$

B

Assuming sinusoidal solutions in time and space, the dispersion relation is

$$-\sigma_m \omega^2 + Sk^2 - \frac{2\varepsilon_0 V_0^2}{d^3} = \pm\frac{\varepsilon_0 V_0^2}{d^3}$$

We have a dispersion relation that factors into two parts. The odd mode, $\xi_1 = -\xi_2$ has the dispersion relation

$$\omega = \left[\frac{Sk^2}{\sigma_m} - \frac{3\varepsilon_0 V_0^2}{\sigma_m d^3}\right]^{\frac{1}{2}}$$

The even mode, $\xi_1 = \xi_2$ has the dispersion relation

$$\omega = \left[\frac{Sk^2}{\sigma_m} - \frac{\varepsilon_0 V_0^2}{\sigma_m d^3}\right]^{\frac{1}{2}}$$

C

A plot of the dispersion relation appears in Figure 1.

D

The lowest allowed value of k is $k = \frac{\pi}{L}$ since the membranes are fixed at $x = 0$ and $x = L$. Therefore the first mode to go unstable is the odd mode. This happens as

$$\left(\frac{3\varepsilon_0 V_0^2}{Sd^3}\right) = \frac{\pi^2}{L^2}$$

or

$$V_0 = \left|\frac{\pi^2}{L^2}\frac{Sd^3}{\varepsilon_0 3}\right|^{\frac{1}{2}}$$

Problem 11.4

We may take the results of Prob. 10.13, replacing $\frac{\partial}{\partial t}$ by $\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}$ and replacing ω by $\omega - kU$.

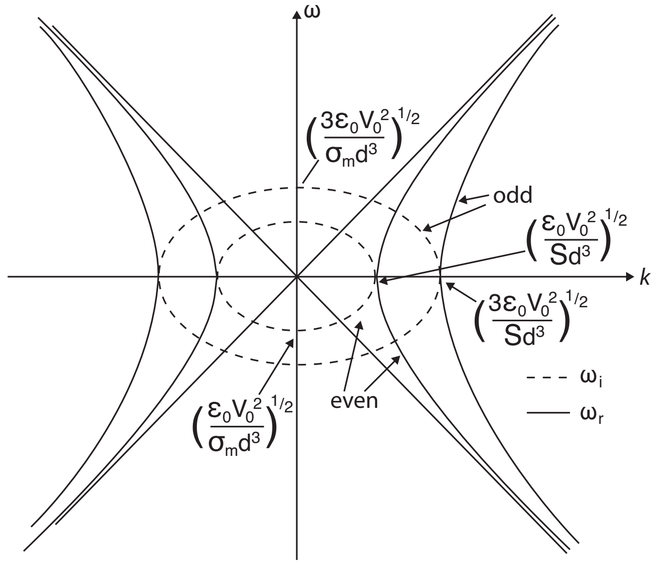

Figure 1: Plot of the dispersion relation for two membranes. (Image by MIT OpenCourseWare.)

A

The equations of motion are

$$\sigma_m \left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)^2 \xi_1 = S\frac{\partial^2 \xi_1}{\partial x^2} + \frac{\varepsilon_0 V_0^2}{d^3}(2\xi_1 - \xi_2) \tag{6}$$

and

$$\sigma_m \left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right) \xi_2 = S\frac{\partial^2 \xi_2}{\partial x^2} + \frac{\varepsilon_0 V_0^2}{d^3}(2\xi_2 - \xi_1) \tag{7}$$

B

The dispersion relation is biquadratic, and factors into

$$-\sigma_m(\omega - kU)^2 + Sk^2 - \frac{2\varepsilon_0 V_0^2}{d^3} = \pm\frac{\varepsilon_0 V_0^2}{d^3} \tag{8}$$

The $(\pm)$ signs correspond to the cases $\xi_1 = -\xi_2$ and $\xi_1 = \xi_2$ respectively, as will be seen in part (d).

C

The dispersion relations are plotted in figures (2) and (3) for $U > \sqrt{\frac{S}{\sigma_m}}$.

D

Let $\xi_1 = \xi_2$. Then (6) and (7) become

$$\sigma_m \left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)^2 \xi_1 = S\frac{\partial^2 \xi_1}{\partial x^2} + \frac{\varepsilon_0 V_0^2}{d^3}\xi_1$$

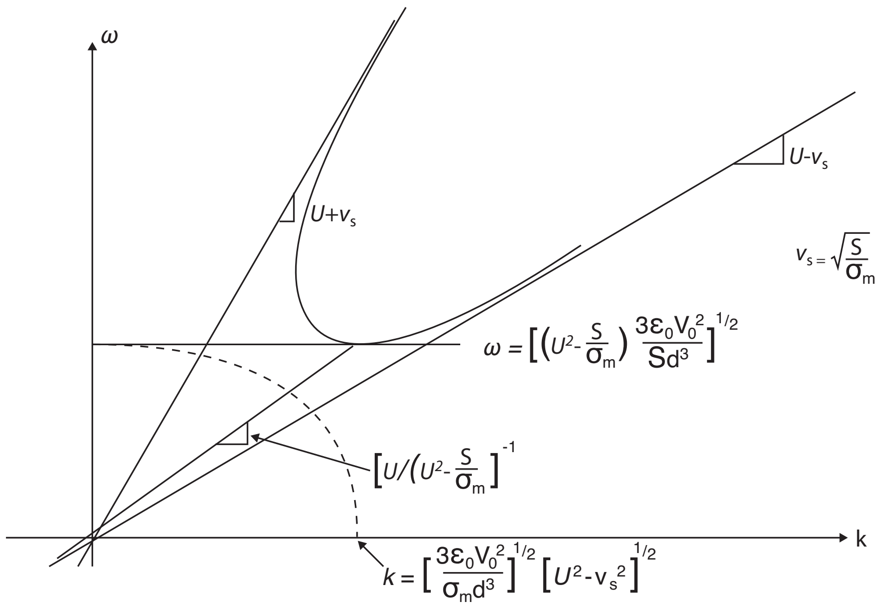

Figure 2: Plot of the dispersion relation for odd motions ($\xi_1 = -\xi_2$). (Image by MIT OpenCourseWare.)

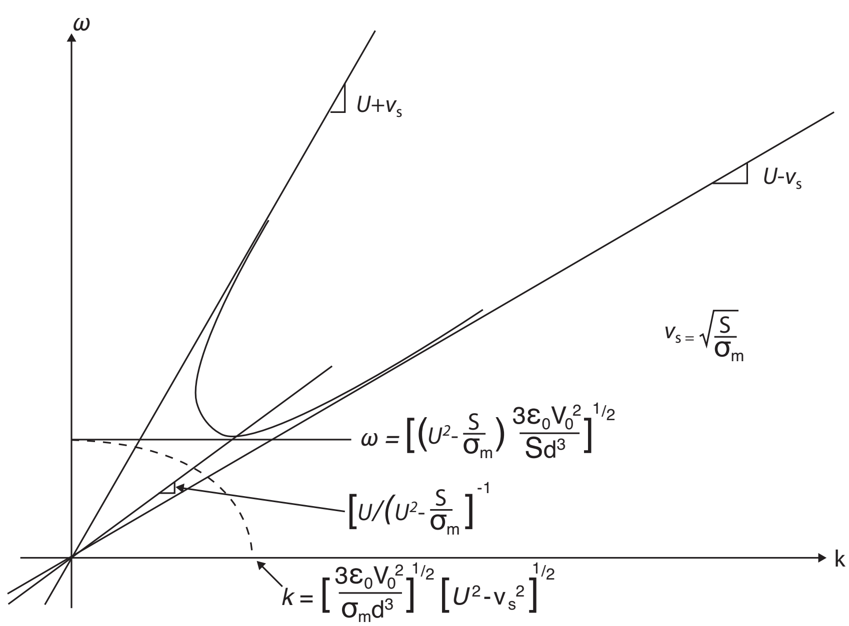

Figure 3: Plot of the dispersion relation for even motions ($\xi_1 = \xi_2$). (Image by MIT OpenCourseWare.)

and

$$\sigma_m \left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)^2 \xi_2 = S\frac{\partial^2 \xi_2}{\partial x^2} + \frac{\varepsilon_0 V_0^2}{d^3}\xi_2$$

These equations are identical for $\xi_1 = -\xi_2$; the dispersion equation is (8) with the + sign.

E

$$\xi_1(0,t) = \text{Re}\hat{\xi}e^{j\omega t} = -\xi_2(0,t)$$
$$\frac{\partial \xi_1}{\partial x} = \frac{\partial \xi_2}{\partial x} = 0 \text{ at } x = 0$$

The odd mode is excited. Hence, we use the + sign in (8)

$$-\sigma_m(\omega - kU)^2 + Sk^2 - \frac{3\varepsilon_0 V_0^2}{d^3} = 0$$

$$k^2(S - \sigma_m U^2) + 2\sigma_m \omega k U - \sigma_m \omega^2 - \frac{3\varepsilon_0 V_0^2}{d^3} = 0$$

Solving for k, we obtain

$$k_\pm = \alpha \pm \beta$$

where $\alpha = \frac{\omega U}{U^2 - v_s^2}$

$$\beta = \frac{\left[\omega^2 v_s^2 - \frac{3\varepsilon_0 V_0^2 (U^2 - v_s^2)}{\sigma_m d^3}\right]^{\frac{1}{2}}}{U^2 - v_s^2}$$

with $v_s^2 = \frac{S}{\sigma_m}$.

Therefore

$$\xi_1 = \text{Re}\left\{\left[Ae^{-j(\alpha+\beta)x} + Be^{-j(\alpha-\beta)x}\right]e^{j\omega t}\right\}$$

Applying the boundary conditions, we obtain

$$A = \hat{\xi}\frac{(\beta - \alpha)}{2\beta}$$

$$B = \frac{(\alpha + \beta)\hat{\xi}}{2\beta}$$

Therefore, if $\hat{\xi}$ is real

$$\xi_1(x,t) = -\xi_2(x,t) = \hat{\xi}\cos\beta x\cos(\omega t - \alpha x) - \frac{\alpha}{\beta}\hat{\xi}\sin\beta x\sin(\omega t - \alpha x)$$

F

We can see that β can be imaginary, for which we will have spatially growing curves. This can happen when

$$\omega^2 v_s^2 - \frac{3\varepsilon_0 V_0^2}{\sigma_m d^3}(U^2 - v_s^2) < 0$$

or

$$V_0^2 > \frac{\sigma_m d^3 \omega^2 v_s^2}{3\varepsilon_0 (U^2 - v_s^2)} \qquad (9)$$

G

With $V_0 = 0$ and $v > v_s$: (see Figure 4)

Amplifying waves are obtained as (9) is satisfied (see Figure 5)

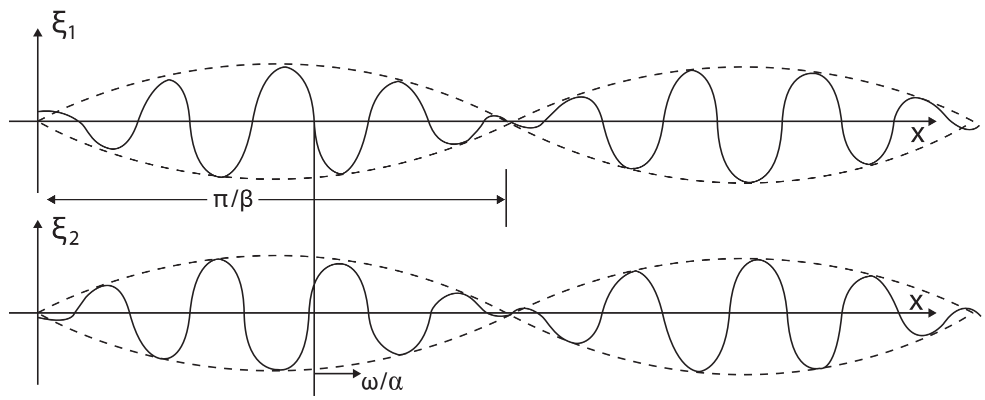

Figure 4: ξ_1 and ξ_2 with $V_0 = 0$ and $v > v_s$. (Image by MIT OpenCourseWare.)

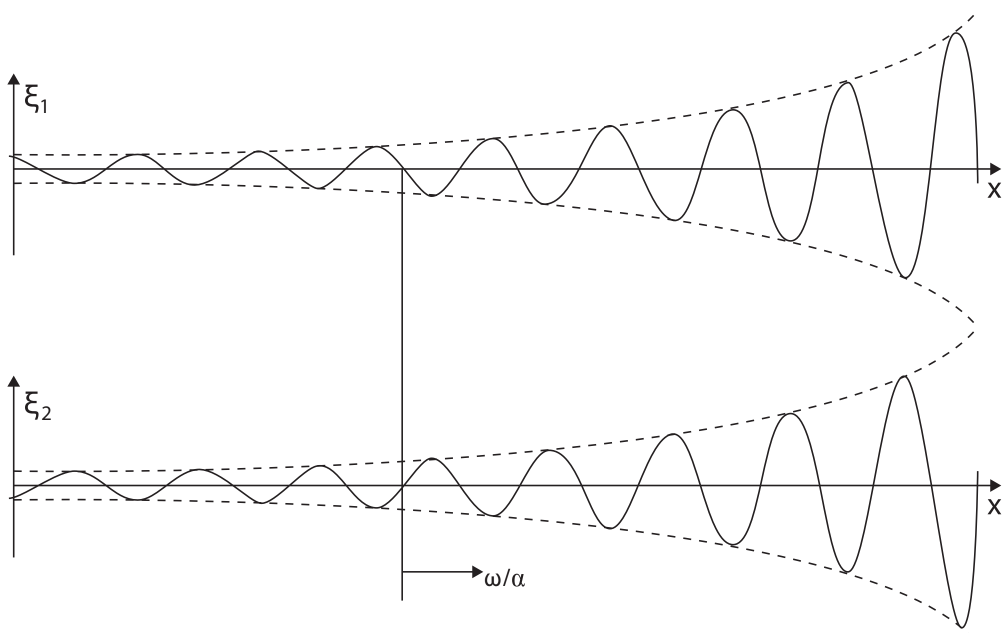

Figure 5: Amplifying waves. (Image by MIT OpenCourseWare.)

Problem 11.5

A

The equation of motion is

$$\sigma_m \left(\frac{\partial}{\partial t} + U \frac{\partial}{\partial x} \right)^2 \xi = S \frac{\partial^2 \xi}{\partial x^2} - \sigma_m g + T \tag{10}$$

with $T = \frac{\varepsilon_0}{2} \frac{V_0^2}{(d-\xi)^2} \approx \frac{\varepsilon_0}{2} V_0^2 \left[\frac{1}{d^2} + \frac{2\xi}{d^3} \right]$.

For equilibrium, $\xi = 0$ and from (10)

$$\frac{\varepsilon_0 V_0^2}{2d^2} = \sigma_m g$$

or

$$V_0 = \left[\frac{2\sigma_m g d^2}{\varepsilon_0}\right]^{\frac{1}{2}}$$

B

With solutions of the form $e^{j(\omega t - kx)}$ the dispersion relation is

$$(\omega - kU)^2 = \frac{S}{\sigma_m}k^2 - \frac{\varepsilon_0 V_0^2}{\sigma_m d^3}$$

Solving for k, we obtain

$$k = \frac{\omega U \pm \sqrt{\frac{S}{\sigma_m}\omega^2 - \left(U^2 - \frac{S}{\sigma_m}\right)\left(\frac{\varepsilon_0 V_0^2}{\sigma_m d^3}\right)}}{(U^2 - \frac{S}{\sigma_m})}$$

For $U > \sqrt{\frac{S}{\sigma_m}}$, and not to have spatially growing waves

$$\frac{S}{\sigma_m}\omega^2 - \left(U^2 - \frac{S}{\sigma_m}\right)\left(\frac{\varepsilon_0 V_0^2}{\sigma_m d^3}\right) > 0$$

or

$$\omega^2 > \left[\left(U^2 - \frac{S}{\sigma_m}\right)\frac{\varepsilon_0 V_0^2}{S d^3}\right]$$