6.641 Electromagnetic Fields, Forces, and Motion
Spring 2005

6.641 — Electromagnetic Fields, Forces, and Motion	Spring 2005
Problem Set 10 - Solutions	
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Problem 10.1

The equation of motion for a static rod is

$$0 = E\frac{\partial^2 \delta}{\partial x^2} + F_x \text{ where } F_x = \rho g$$

We can integrate this equation directly and get

$$\delta(x) = -\frac{\rho g}{E}\left(\frac{x^2}{2}\right) + Cx + D$$

where C and D are arbitrary constants.

A

The stress function is $T(x) = E\frac{d\delta}{dx}$, and therefore

$$T(x) = -\rho g x + CE$$

We have a free end at $x = l$ and this implies $T(x = l) = 0$. Now we can write the stress as

$$T(x) = -\rho g x + \rho g l$$

The maximum stress occurs at $x = 0$ and is $T_{\text{max}} = \rho g l$. Equating this to the maximum allowable stress, we have

$$2 \times 10^9 = (7.8 \times 10^3)(9.8)l$$

hence

$$l = 2.6 \times 10^4 \text{ meters}$$

B

From part (a)

$$T(x) = -\rho g x + \rho g l$$

The fixed end at $x = 0$ implies that $D = 0$, so now we can write the displacement

$$\delta(x) = -\frac{\rho g}{E}\left(\frac{x^2}{2}\right) + \frac{\rho g l}{E}(x)$$

C

$$\delta(l) = -\frac{\rho g}{E}\frac{l^2}{2} + \frac{\rho g l}{E}(l) = \frac{\rho g l^2}{2E}$$

For $l = 2.6 \times 10^4$ meters, $\delta(l) = 129$ meters. This appears to be a large displacement, but note that the total unstressed length is 26,000 meters.

Problem 10.2

From the characteristic equations

$$\rho\frac{\partial v}{\partial t} = \frac{\partial T}{\partial x}, \qquad v = \frac{C_+ + C_-}{2}$$

$$\frac{\partial T}{\partial t} = E\frac{\partial v}{\partial x}, \qquad \frac{T}{\sqrt{\rho E}} = \frac{C_- - C_+}{2}$$

$$v + \frac{1}{\sqrt{\rho E}}T = C_-$$

$$v - \frac{1}{\sqrt{\rho E}}T = C_+$$

A

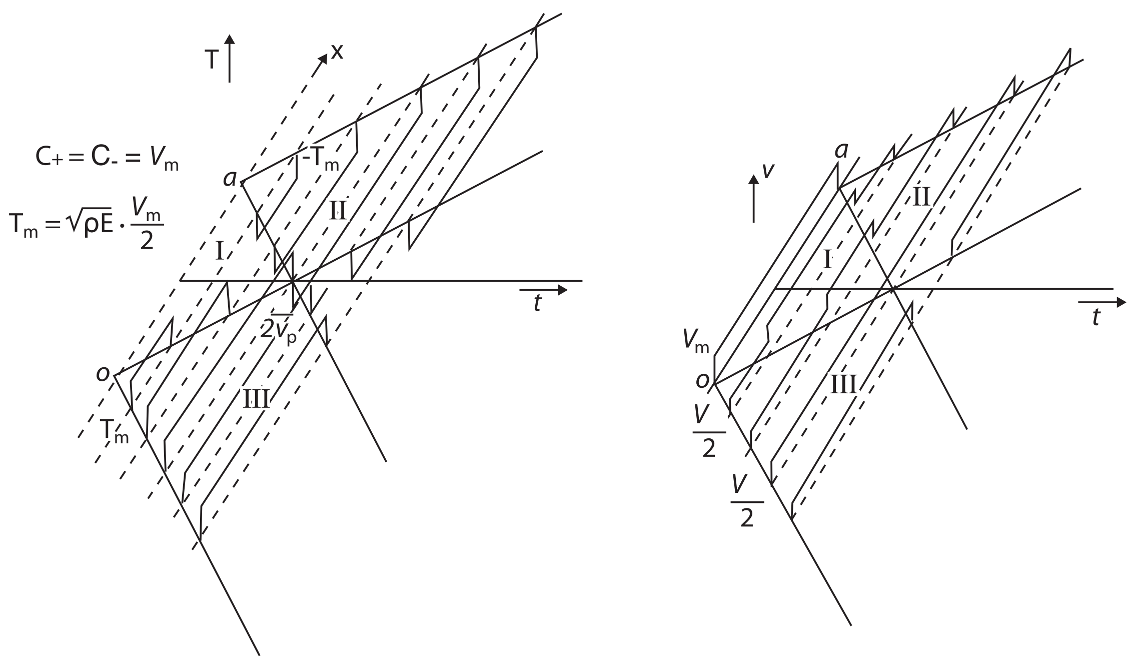

Figure 1: Tension and medium velocity in $x - t$ space for an infinite extent elastic medium (Image by MIT OpenCourseWare.)

$$I : C_+ = C_- = v_m$$

$$II : C_+ = v_m, C_- = 0$$

$$III : C_+ = 0, C_- = v_m$$

B

$$I : C_+ = C_- = v_m$$
$$II : C_+ = v_m, C_- = -v_m$$
$$III : C_+ = -v_m, C_- = v_m$$
$$IV : C_+ = -v_m, C_- = -v_m$$
$$V : C_+ = -v_m, C_- = v_m$$
$$VI : C_+ = v_m, C_- = -v_m$$

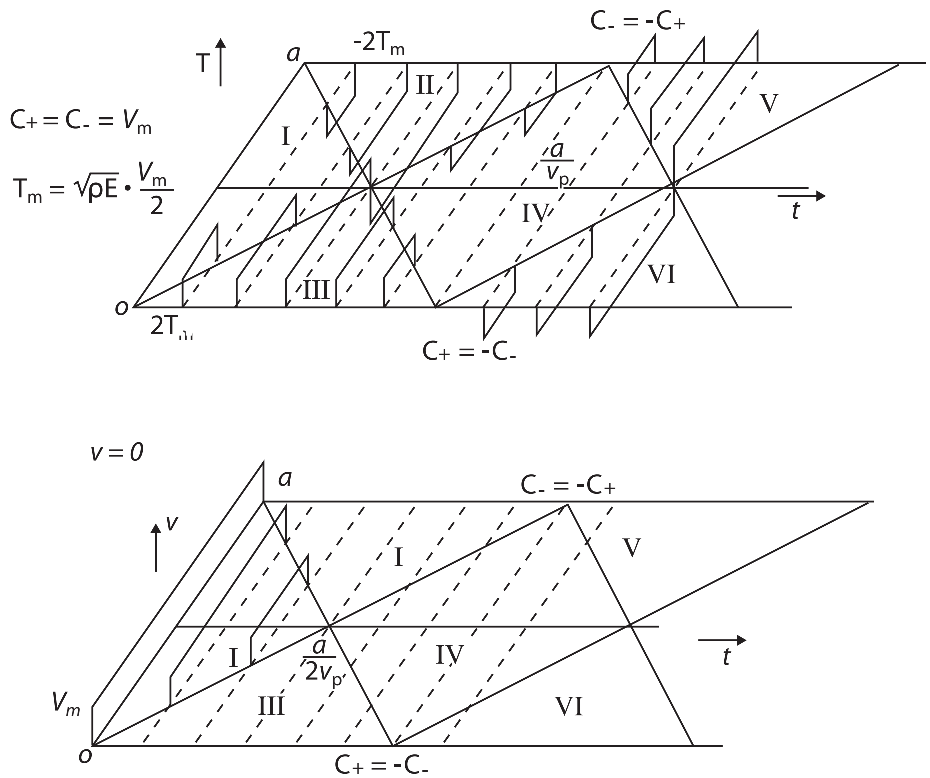

Figure 2: Tension and medium velocity in $x - t$ space for an elastic rod of length a. (Image by MIT OpenCourseWare.)

At $x = 0$, $x = l$ fixed boundary $v = 0$

$$C_- = -C_+$$

Problem 10.3

First, we can calculate the force of magnetic origin, f_x, on the rod. If we define $\delta(l, t)$ to be the a.c. deflection on the rod at $x = l$, then using Ampere's law and the Maxwell stress tensor (Eq. 8.5.41 with magnetostriction ignored) we find

$$f_x = \frac{\mu_0 A N^2 I^2}{2\left(d - \delta(l, t)\right)^2}$$

This result can also be obtained using the energy methods of Chap. 3 (See Appendix E, Table 3.1). Since $d \gg \delta(l,t)$, we may linearize f_x

$$f_x = \frac{\mu_0 A N^2 I^2}{2d^2} + \frac{\mu_0 A N^2 I^2}{d^3}\delta(l,t)$$

The first term represents a constant force which is balanced by a static deflection on the rod. If we assume that this static deflection is included in the equilibrium length l, then we need only use the last term of f_x to compute the dynamic deflection $\delta(l,t)$. In the bulk of the rod we have the wave equation; for sinusoidal variations

$$\delta(x,t) = \text{Re}\left[\hat{\delta}(x)e^{j\omega t}\right]$$

we can write the complex amplitude $\hat{\delta}(x)$ as

$$\hat{\delta}(x) = C_1 \sin\beta x + C_2 \cos\beta x$$

where $\beta = \omega\sqrt{\frac{\rho}{E}}$. At $x=0$ we have a fixed end, so $\hat{\delta(0)} = 0$ and $C_2 = 0$. At $x = l$ the boundary condition is

$$0 = f_x - AE\frac{\partial\delta}{\partial x}(l,t)$$

or

$$0 = \frac{\mu_0 A N^2 I^2}{d^3}\hat{\delta}(x=l) - AE\frac{d\hat{\delta}}{dx}(x=l)$$

Substituting we obtain

$$\frac{\mu_0 A N^2 I^2}{d^3}C_1 \sin\beta l = C_1 AE\beta\cos\beta l \tag{1}$$

Our solution is $\hat{\delta}(x) = C_1\sin\beta x$ and for a non-trivial solution we must have $C_1 \neq 0$. So, divide (1) by C_1 to obtain the resonance condition:

$$\left(\frac{\mu_0 A N^2 I^2}{d^3}\right)\sin\beta l = AE\beta\cos\beta l$$

Substituting $\beta = \sqrt{\frac{\rho}{E}}$ and rearranging, we have

$$\frac{Ed^3}{\mu_0 N^2 I^2 l}\left(\omega l\sqrt{\frac{\rho}{E}}\right) = \tan\left(\omega l\sqrt{\frac{\rho}{E}}\right) \tag{2}$$

which, when solved for ω, yields the eigenfrequencies. Graphically, the first two eigenfrequencies are found from the sketch. Notice that as the current I is increased, the slope of the straight line decreases and the first eigenfrequency (denoted by ω_1) goes to zero and then seemingly disappears for still higher currents. Actually ω_1 now becomes imaginary and can be found from the equation

$$\frac{Ed^3}{\mu_0 N^2 I^2 l}\left(|\omega_1| l\sqrt{\frac{\rho}{E}}\right) = \tanh\left(|\omega_1| l\sqrt{\frac{\rho}{E}}\right)$$

Just as there are negative solutions to (2), $-\omega_1, -\omega_2, \ldots$ etc., so there are now solutions $\pm j|\omega_1|$. Thus, because ω_1 is imaginary, the system is unstable, (amplitude of one solution growing in time).

Hence when the slope of the straight line becomes less than unity, the system is unstable. This condition can be stated as

$$\text{STABLE} \longrightarrow \frac{Ed^3}{\mu_0 N^2 I^2 l} > 1$$

$$\text{UNSTABLE} \longrightarrow \frac{Ed^3}{\mu_0 N^2 I^2 l} < 1$$

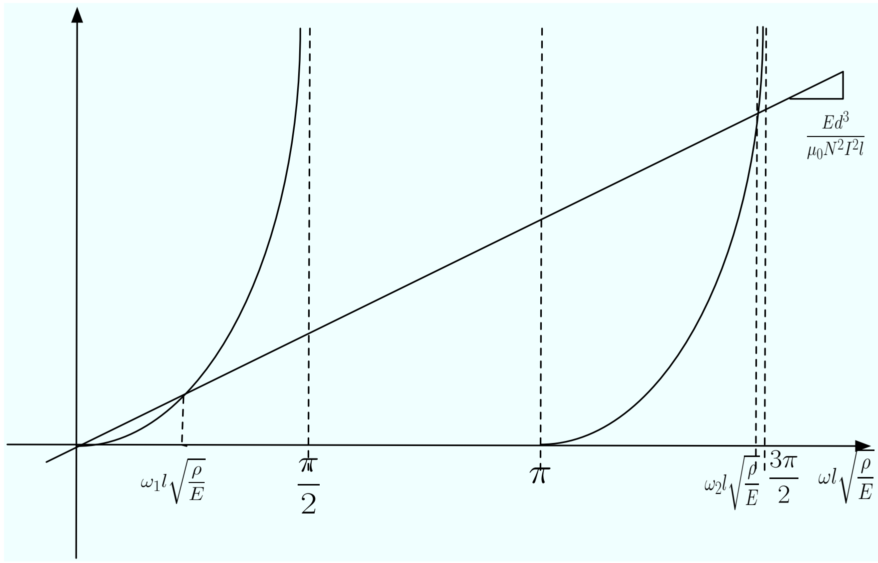

Figure 3: Sketch used to find eigenfrequencies in Problem 10.3. (Image by MIT OpenCourseWare.)

Problem 10.4

A

At the outset, we can write the equation of motion for the massless plate:

$$-aT(l,t) + f^e(t) = M\frac{\partial^2\delta}{\partial t^2}(l,t) \approx 0$$

Using the maxwell stress tensor we find the force of electrical origin $f^e(t)$ to be

$$f^e(t) = \frac{\varepsilon_0 A}{2}\left[\frac{(V_0+v(t))^2}{(d-\delta(l,t))^2} - \frac{(V_0-v(t))^2}{(d-\delta(l,t))^2}\right]$$

Since $v(t) \ll V_0$ and $\delta(l,t) \ll d$, we can linearize $f^e(t)$:

$$f^e(t) = \left[\frac{2\varepsilon_0 A V_0^2}{d^3}\right]\delta(l,t) + \left[\frac{2\varepsilon_0 A V_0}{d^2}\right]v(t)$$

Recognizing that $T(l,t) = E\frac{\partial\delta}{\partial x}(l,t)$ we can write our boundary condition at $x = l$ in the desired form

$$aE\frac{\partial\delta}{\partial x}(l,t) = \frac{2\varepsilon_0 A V_0^2}{d^3}\delta(l,t) + \frac{2\varepsilon_0 A V_0}{d^2}v(t)$$

Longitudinal displacements in the rod obey the wave equation and for an assumed form of $\delta(x,t) = \text{Re}\left[\hat{\delta}(x)e^{j\omega t}\right]$ we can write $\hat{\delta}(x) = C_1\sin\beta x + C_2\cos\beta x$, where $\beta = \omega\sqrt{\frac{\rho}{E}}$. At $x = 0$ we have a fixed end, thus $\hat{\delta}(x=0) = 0$ and $C_2 = 0$. From part (a) and assuming sinusoidal time dependence, we can write our boundary condition at $x = l$ as

$$aE\frac{d\hat{\delta}}{dx}(l) = \frac{2\varepsilon_0 A V_0^2}{d^3}\hat{\delta}(l) + \frac{2\varepsilon_0 A V_0}{d^2}\hat{V}$$

Solving

$$C_1 = \frac{2\varepsilon_0 A V_0 \hat{V}}{aEd^2\beta\cos\beta l - \frac{2\varepsilon_0 A V_0^2}{d}\sin\beta l}$$

Finally, we can write our solution as

$$\delta(x,t) = \left[\frac{2\varepsilon_0 A V_0 \sin\beta x}{aEd^2\beta\cos\beta l - \frac{2\varepsilon_0 A V_0^2}{d}\sin\beta l}\right] \mathrm{Re}\left[\hat{V} e^{j\omega t}\right]$$

Problem 10.5

A

$$i(z,t) = \frac{C}{\Delta z}\frac{\partial}{\partial t}\left[v(z-\Delta z) - v(z)\right]; \qquad v(z,t) = \frac{L}{\Delta z}\frac{\partial}{\partial t}\left[i(z) - i(z+\Delta z)\right]$$

$$\lim_{z\to 0} i(z,t) = -C\frac{\partial^2 v}{\partial t \partial z}; \qquad v(z,t) = -L\frac{\partial^2 i}{\partial t \partial z}$$

B

$$i(z,t) = \mathrm{Re}\hat{i}e^{j(\omega t - kz)}, \qquad v(z,t) = \mathrm{Re}\hat{v}e^{j(\omega t - kz)}$$

$$\hat{i} = -C\omega k\hat{v}; \qquad \hat{v} = -L\omega k\hat{i}$$

$$\hat{i} = +LC\omega^2 k^2 \hat{i} \rightarrow LC\omega^2 k^2 = 1 \rightarrow k = \pm\frac{1}{\omega\sqrt{LC}}$$

C

$$v_p = \frac{\omega}{k} = \omega^2\sqrt{LC}$$

$$v_g = \frac{d\omega}{dk} = -\omega^2\sqrt{LC}$$

Such systems are called backward wave because the group velocity is opposite in direction to the phase velocity.

D

$$\hat{v}(z) = V_1 \sin kz + V_2 \cos kz$$

$$\hat{v}(z=0) = 0 = V_2$$

$$\hat{v}(z=-l) = V_0 = -V_1 \sin kl \rightarrow \hat{v}(z) = \frac{-V_0}{\sin kl}\sin kz$$

$$\hat{i}(z) = -Cj\omega\frac{d\hat{v}}{dz} = \frac{j\omega C V_0 k \cos kz}{\sin kl} = j\sqrt{\frac{C}{L}}V_0\frac{\cos kz}{\sin kl}$$

E

$$\text{Resonance } \rightarrow \sin kl = 0 \rightarrow kl = n\pi \rightarrow \omega_n = \frac{1}{\left(\frac{n\pi}{l}\right)\sqrt{LC}}$$

Problem 10.6

A

$$v(t=0) = \frac{V_0 R_L}{R_L + R_s} = V_+ + V_- \qquad V_+ = \frac{V_0}{2}\frac{(R_L + Z_0)}{(R_L + R_s)}$$

$$i(t=0) = \frac{V_0}{R_L + R_s} = Y_0(V_+ - V_-) \qquad V_- = \frac{V_0}{2}\frac{(R_L - Z_0)}{(R_L + R_S)}$$

B

$$V_{+n} = A(\Gamma_s \Gamma_L)^n; \Gamma_s = \frac{R_s - Z_0}{R_s + Z_0}, \Gamma_L = \frac{R_L - Z_0}{R_L + Z_0}$$

$$V_{-n} = \Gamma_L V_{+n} = A\Gamma_L(\Gamma_S \Gamma_L)^n$$

$$V_{+n=0} = A = \frac{V_0}{2}\left(\frac{R_L + Z_0}{R_L + R_s}\right)$$

$$V_n = V_{+n} + V_{-n} = \frac{V_0}{2}\left(\frac{R_L + Z_0}{R_L + R_s}\right)\left[1 + \frac{R_L - Z_0}{R_L + Z_0}\right](\Gamma_s \Gamma_L)^n = \frac{V_0 R_L}{R_L + R_s}(\Gamma_s \Gamma_L)^n$$