6.641 Electromagnetic Fields, Forces, and Motion
Spring 2005

6.641 — Electromagnetic Fields, Forces, and Motion	Spring 2005
Problem Set 9 - Solutions	
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Problem 9.1

A

$$\nabla \times \bar{E} = -\frac{\partial \bar{B}}{\partial t}$$
$$\nabla \times \bar{B} = \mu\sigma E$$

So

$$\nabla \times \nabla \times \bar{B} = -\mu\sigma\frac{\partial \bar{B}}{\partial t}$$

But

$$\nabla \times (\nabla \times \bar{B}) = \nabla(\nabla \cdot \bar{B}) - \nabla^2\bar{B} = -\nabla^2\bar{B}$$

So

$$\nabla^2\bar{B} = \mu\sigma\frac{\partial \bar{B}}{\partial t}$$

B

Since $\bar{B}$ only has a z component

$$\nabla^2 B_z = \mu\sigma\frac{\partial B_z}{\partial t}$$

In cylindrical coordinates

$$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} + \frac{\partial^2}{\partial z^2}$$

Here $B_z = B_z(r,t)$, so

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \hat{B}}{\partial r}\right) + \mu\sigma\alpha\hat{B} = 0$$

C

We want the magnetic field to remain finite at $r = 0$, hence $C_2 = 0$.

D

At $r = a$

$$B(a,t) = \mu_0 H_0 - C_1 J_0(\sqrt{\mu_0\sigma\alpha}a) = \mu_0 H_0$$

Hence if $C_1 \neq 0$

$$J_0(\sqrt{\mu_0\sigma\alpha}a) = 0$$

E

Multiply both sides of expression for $B(r, t = 0) = 0$ by $rJ_0(v_j \frac{r}{a})$ and integrate from 0 to a. Then,

$$\int_0^a \mu_0 H_0 r J_0(v_j \frac{r}{a}) dr = \mu_0 H_0 \frac{a^2}{v_j} J_1(v_j)$$

$$\int_0^a \sum_{i=1}^{\infty} C_i J_0(v_i \frac{r}{a}) r J_0(v_j \frac{r}{a}) dr = C_j \frac{a^2}{2} J_1^2(v_j)$$

from which it follows that

$$C_j = \frac{2\mu_0 H_0}{v_j J_1(v_j)}$$

The values of v_j and $J_1(v_j)$ given in the table lead to the coefficients

$$\frac{C_1}{2\mu_0 H_0} = .802; \frac{C_2}{2\mu_0 H_0} = -.535; \frac{C_3}{2\mu_0 H_0} = .425$$

F

$$\alpha_1 = \frac{1}{\mu_0 \sigma} \left(\frac{v_1}{a} \right)^2$$

$$\tau_1 = \frac{\mu_0 \sigma a^2}{v_1^2} = 0.174 \mu_0 \sigma a^2$$

$$\tau_1 = (0.174)(4\pi \times 10^{-7}) \frac{10^4}{4\pi} (25) \times 10^{-4}$$

$$\approx 4.35 \times 10^{-7} \text{ seconds}$$

Problem 9.2

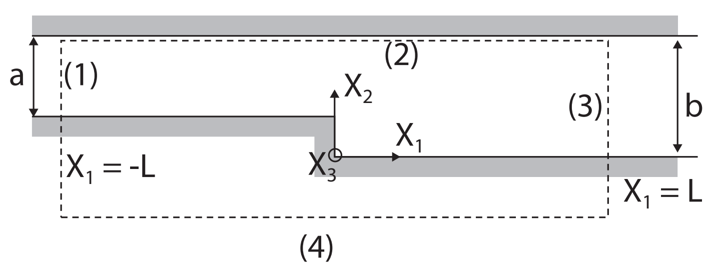

Figure 1: Diagram of surfaces (1), (2), (3), and (4) to evaluate the force on the lower plate using the Maxwell Stress Tensor. (Image by MIT OpenCourseWare.)

Before finding the force, we must calculate the $\bar{H}$ field at $x_1 = L$. To find this field let us use

$$\oint \bar{B} \cdot \bar{n} da = 0 \tag{1}$$

over the dotted surface. At $x_1 = +L$,

$$\bar{H}(x_1 = L) = H_0 \bar{i}_1$$

over surface (4) $\bar{H} = 0$, and over surface (2), $\bar{H}$ is in the $\bar{i}_1$ direction, where $\bar{n} = \bar{i}_2$. Thus over surface (2), $\bar{B} \cdot \bar{n} = 0$. Hence, the integral in (1) reduces to

$$-\int_{(1)} \mu_0 H_0 da + \int_{(3)} \mu_0 H(x_1 = +L) da = 0$$

$$-\mu_0 H_0 a + \mu_0 H b = 0 \qquad \text{per unit length}$$

Thus:

$$\bar{H}(x_1 = +L) = (a/b) H_0 \bar{i}_1$$

$$T_{ij} = \mu_0 H_i H_j - \frac{\delta_{ij}}{2} \mu_0 H_k H_k$$

Hence, the stress tensor over surfaces (1), (2), and (3) is:

$$T_{ij} = \begin{bmatrix} \frac{\mu_0}{2} H_1^2 & 0 & 0 \\ 0 & -\frac{\mu_0}{2} H_1^2 & 0 \\ 0 & 0 & -\frac{\mu_0}{2} H_1^2 \end{bmatrix}$$

Over surface (4)

$$T_{ij} = [0]$$

Thus the force in the 1 direction is

$$f_1 = \int T_{ij} n_j \cdot da$$

$$f_1 = -\int_{(1)} T_{11} da + \int_{(3)} T_{11} da + \int_{(2)} T_{12} da$$

Thus, since the last integral makes no contribution,

$$f_1 = -\frac{\mu_0}{2} H_0^2 (a) + \frac{\mu_0}{2} H_0^2 \left(\frac{a}{b}\right)^2 \cdot b = \frac{\mu_0}{2} H_0^2 a \left(\frac{a}{b} - 1\right) \tag{2}$$

Since $T_{ij} = 0$ over surface (4) there is no contribution to the force from this surface and, by symmetry, there is no contribution to the force from the surfaces perpendicular to the x_j axis. Thus, the force per unit depth in 1 direction is (2).

Problem 9.3

First, let us note the $\bar{E}$ fields on each of the surfaces of the figure over surfaces (1), (3), (5), and (7), $E_1 = 0$.
Over surface

(6) $E_2 = \dfrac{V_0}{a}$ $\qquad E_1 = 0$

(4) $E_2 = \dfrac{V_0}{b}$ $\qquad E_1 = 0$

(2) $E_2 = \dfrac{V_0}{c}$ $\qquad E_1 = 0$

From Eq. 8.3.10,

$$T_{ij} = \varepsilon_0 E_i E_j - \frac{\delta_{ij}}{2} \varepsilon_0 E_k E_k$$

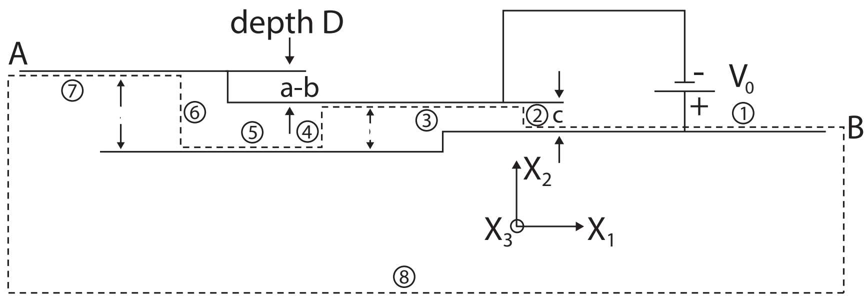

Figure 2: Diagram of surfaces (1)-(8) used to find force on the lower electrode using the Maxwell Stress Tensor. (Image by MIT OpenCourseWare.)

Hence, over surfaces (1), (3), (5) and (7)

$$T_{12} = 0 \tag{3}$$

and over surfaces

$$(6) \quad T_{11} = -\frac{\varepsilon_0}{2}\left(\frac{V_0}{a}\right)^2$$

$$(4) \quad T_{11} = -\frac{\varepsilon_0}{2}\left(\frac{V_0}{b}\right)^2$$

$$(2) \quad T_{11} = -\frac{\varepsilon_0}{2}\left(\frac{V_0}{c}\right)^2$$

Now

$$f_1 = \int T_{ij} n_j da = \int T_{11} n_1 da + \int T_{12} n_2 da + \int T_{13} n_3 da$$

$$\int T_{13} n_3 da = 0 \quad \text{because the problem is two dimensional}$$

Let us consider each of the other integrals:

$$\int T_{12} n_2 da = 0$$

because the surfaces that have normal n_2 are (1),(3),(5), and (7) and by (3) we have shown that $T_{12} = 0$ over these surfaces. Also, we get no contribution to the force over surface (8), because $\bar{E} \to 0$ faster than the area $\to \infty$. Hence the calculation of the force reduces to

$$f_1 = \int_{(6)} T_{11}^{(6)} da_6 - \int_{(4)} T_{11}^{(4)} da_4 - \int_{(2)} T_{11}^{(2)} da_2$$

$$f_1 = -\frac{\varepsilon_0 D V_0^2}{2}\left(\frac{1}{a} - \frac{1}{b} + \frac{1}{c}\right)$$

Note: by symmetry, there is no contribution to the force from the surfaces perpendicular to the x_3 axis.

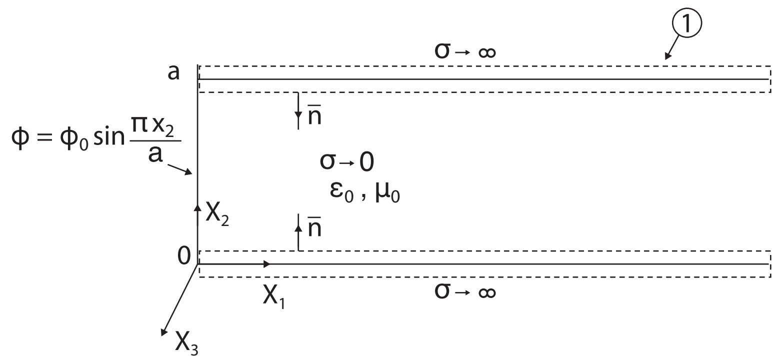

Figure 3: Diagram of grounded electrodes and distributed electric potential source at $x_1 = 0$. (Image by MIT OpenCourseWare.)

Problem 9.4

A

From elementary field theory, we find that

$$\phi = \phi_0 \sin \frac{\pi x_2}{a} e^{-\frac{\pi x_1}{a}}$$

satisfies $\nabla^2 \phi = 0$ in the region between the plates and the required boundary conditions. The distribution of $\bar{E}$ follows from

$$\bar{E} = -\nabla \phi$$

Hence,

$$\bar{E} = \frac{\pi \phi_0}{a} e^{-\frac{\pi x_1}{a}} \left[\sin \frac{\pi x_2}{a} \bar{i}_1 - \cos \frac{\pi x_2}{a} \bar{i}_2 \right]$$

The sketch of the $\bar{E}$ field is obtained by recognizing that $\bar{E}$ is directed perpendicular to contours of constant ϕ.

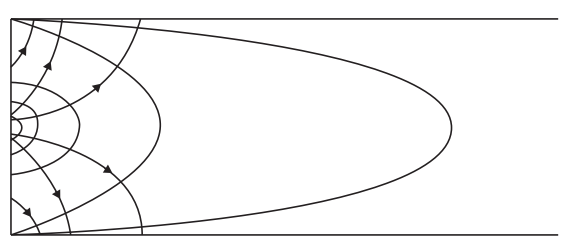

Figure 4: Sketch of the $\bar{E}$ field and equipotential lines. (Image by MIT OpenCourseWare.)

B

To find the force at the bottom plate, we use surface (2). $\bar{E} = 0$ everywhere except on the upper side where the normal $\bar{n} = \bar{i}_2$ and the field is

$$\bar{E} = -\frac{\pi \phi_0}{a} e^{-\frac{\pi x_1}{a}} \bar{i}_2$$

Hence,

$$f_1 = \int T_{ij} n_j da = 0$$

$$f_2 = \int T_{2j} n_j da = \int T_{22} n_2 da_2$$

per unit x_3. This reduces to

$$f_2 = \int_0^\infty T_{22} dx_1$$

but, $T_{22} = \frac{1}{2}\varepsilon_0 E_2 E_2 = \frac{1}{2}\varepsilon_0 \frac{\pi^2 \phi_0^2}{a^2} e^{-\frac{2\pi x_1}{a}}$ and thus

$$f_2 = \frac{\varepsilon_0 \pi^2 \phi_0^2}{2a^2} \int_0^\infty e^{-\frac{2\pi x_1}{a}} dx_1$$

$$f_2 = \frac{\varepsilon_0 \pi \phi_0^2}{4a}$$

C

On the top plate, use surface (1). Only the sign of the normal changes, and the result is

$$f_1 = 0$$

$$f_2 = -\frac{\varepsilon_0 \pi \phi_0^2}{4a}$$

or the force is equal and opposite to that on the bottom plate.

Problem 9.5

A

Since $\overline{J'} = \overline{J}$

$$\begin{aligned} \overline{K} &= \overline{\mathbf{i_z}} K_0 \cos(kUt - kx) \\ &= \overline{\mathbf{i_z}} K_0 \cos(\omega t - kx); \quad \omega = kU \end{aligned}$$

B

The track can be taken as large in the y direction when it is many skin depths thick

$$L = \text{ track thickness } \gg \delta = \sqrt{\frac{2}{\omega \mu_0 \sigma}} = \sqrt{\frac{2}{kU\mu_0\sigma}}$$

In the track we have the diffusion equation

$$\frac{1}{\mu_0 \sigma} \nabla^2 \overline{B} = \frac{\partial \overline{B}}{\partial t}$$

or, with $\overline{B} = \text{Re}\hat{\overline{B}} \exp j(\omega t - kx)$,

$$\frac{1}{\mu_0 \sigma} \left(\frac{\partial^2 \hat{B}_x}{\partial y^2} - k^2 \hat{B}_x \right) = j\omega \hat{B}_x$$

Let $\hat{B}_x(y) = Ce^{\alpha y}$, then

$$\frac{1}{\mu_0\sigma}\alpha^2 = j\omega + \frac{k^2}{\mu_0\sigma}$$

$$\alpha = k\sqrt{1+jS}; \quad S = \frac{\omega\mu_0\sigma}{k^2} = \frac{U\mu_0\sigma}{k}$$

Since the track is modeled as infinitely thick

$$B_x = Ce^{\alpha y}e^{j(\omega t - kx)}$$

The gap between track and train is very thin; thus,

$$-\bar{i}_y \times \frac{\overline{B}}{\mu_0} = \overline{K} = K_0 e^{j(\omega t - kx)}\overline{i_z}$$

which yields

$$B_x(x, y, t) = \mu_0 K_0 e^{\alpha y} e^{j(\omega t - kx)}$$

We must also have $\nabla \cdot \overline{B} = \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} = 0$ or

$$B_y = \frac{jk}{\alpha} B_x(x, y, t)$$

To compute the current in the track we note that

$$\nabla \times \overline{B} = \overline{\mathbf{i_z}}\left(\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y}\right) = \mu_0 \overline{J}$$

$$\overline{J} = -\left(j\frac{S}{\alpha}k^2\right)\frac{B_x}{\mu_0}(x, y, t)\overline{\mathbf{i_z}}$$

C

The time average force density in the track is

$$\langle F_y \rangle = \frac{1}{2}\mathrm{Re}(J_z B_x^*)$$

Hence the time average lifting force per unit $x - z$ area on the train is

$$\begin{aligned}
\langle T_y \rangle &= -\int_{-\infty}^{0} \langle F_y \rangle \, dy = -\mathrm{Re}\int_{-\infty}^{0} \frac{1}{2} J_z B_x^* dy \\
&= \frac{1}{4}\mu_0 K_0^2 \left(\frac{\sqrt{1+S^2} - 1}{\sqrt{1+S^2}}\right) > 0
\end{aligned}$$

D

The time average force density in the track in the x direction is

$$\langle F_x \rangle = -\frac{1}{2}\mathrm{Re}(J_z B_y^*)$$

The force on the train in the x direction is then

$$\begin{aligned}
\langle T_x \rangle &= -\int_{-\infty}^{0} \langle F_x \rangle \, dy = \frac{1}{2}\mathrm{Re}\int_{-\infty}^{0} J_z B_y^* dy \\
&= -\frac{\mu_0 K_0^2}{4}\frac{S}{\sqrt{1+S^2}\,\mathrm{Re}\sqrt{1+jS}} < 0
\end{aligned}$$

The problem is that this force drags the train instead of propelling it in the x direction. To make matters worse, if the train stops, the magnetic levitation force becomes zero.

Problem 9.6

A

From Eq. 8.1.11,

$$T_{ij} = \begin{bmatrix} \frac{1}{2\mu_0}(B_x^2 - B_y^2) & \frac{B_x B_y}{\mu_0} & 0 \\ \frac{B_x B_y}{\mu_0} & \frac{1}{2\mu_0}(-B_x^2 + B_y^2) & 0 \\ 0 & 0 & \frac{1}{2\mu_0}(-B_x^2 - B_y^2) \end{bmatrix}$$

where the components of $\bar{B}$ are given in the problem.

B

The appropriate surface of integration, which is fixed with respect to the fixed frame, is shown in Figure (5). We compute the time average force, and hence contributions from surfaces (1) and (3) cancel. Fields go to zero on surface (2), which is at $y \to \infty$. Thus, there remains the stress on surface (4). The time average value of the surface force density $\bar{T}$ is independent of x. Hence,

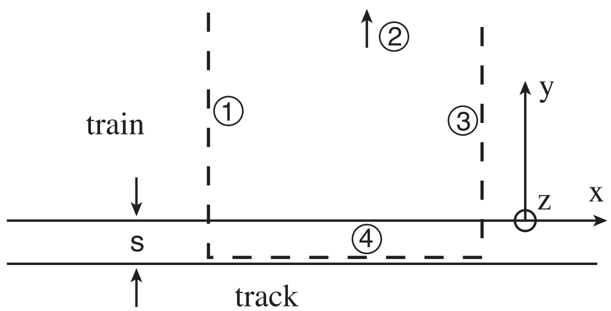

Figure 5: Diagram of the Maxwell Stress Tensor surface to find the levitation force on a train. (Image by MIT OpenCourseWare.)

$$T_y = - < T_{yy}(y = 0) >$$

$$T_y = -\frac{1}{2\mu_0} < -B_x^2 + B_y^2 > \tag{4}$$

Observe that

$$\left\langle \mathrm{Re}\hat{A}e^{-jkUt}\mathrm{Re}\hat{B}e^{-jkUt} \right\rangle \equiv \frac{1}{2}\mathrm{Re}\hat{A}\hat{B}^*$$

where $\hat{B}^*$ is complex conjugate of $\hat{B}$, and (4) becomes

$$T_y = -\frac{1}{4\mu_0}\mathrm{Re}\left\{ -(\mu_0 K_0 e^{jkx})(\mu_0 K_0 e^{-jkx}) + \frac{(-jk\mu_0 K_0)}{\alpha} e^{jkx} \frac{(jk\mu_0 K_0)}{\alpha^*} e^{-jkx} \right\}$$

$$= \frac{\mu_0 K_0^2}{4}\left(1 - \frac{k^2}{\alpha\alpha^*}\right) \tag{5}$$

Finally, use the given definition of α to write (5) as

$$T_y = \frac{\mu_0 K_0^2}{4}\left[1 - \frac{1}{\sqrt{1+\left(\frac{\mu_0\sigma U}{k}\right)^2}}\right]$$

Note that T_y is positive so that the train is supported by the magnetic field. However, as $U \to 0$ (the train is stopped) the levitation force goes to zero.

C

For the force per unit area in the x direction

$$T_x = -\frac{1}{2\mu_0} < B_x B_y(y=0) >$$

$$= -\frac{1}{2\mu_0}\text{Re}\left[\mu_0 K_0 e^{jkx}\left(\frac{jk\mu_0}{\alpha^*}\right) K_0 e^{-jkx}\right]$$

Thus

$$T_x = -\frac{\mu_0 K_0^2}{2\left(1+\left(\frac{\mu_0\sigma U}{k}\right)^2\right)^{\frac{1}{2}}}\text{Re}\, j\sqrt{1 - j\left(\frac{\mu_0\sigma U}{k}\right)} \qquad (6)$$

As must be expected, the force on the train in the x direction vanishes as $U \to 0$. Note that in any case the force always tends to retard the motion and hence could hardly be used to propel the train.

The identity $\sin(\theta/2) = \pm\sqrt{(1-\cos\theta)/2}$ is helpful in reducing (6) to the form

$$T_x = \frac{-\mu_0 K_0^2}{2\left[1+\left(\frac{\mu_0\sigma U}{k}\right)^2\right]^{\frac{1}{2}}}\sqrt{\frac{1}{2}\left(\sqrt{1+\left(\frac{\mu_0\sigma U}{k}\right)^2} - 1\right)}$$

Problem 9.7

A

From Ampere's Law,

$$B_z = \frac{\mu_0 N i_F}{D}$$

B

$$\lambda = NWTB_z \equiv Li_F \Rightarrow L = \frac{\mu_0 N^2 WT}{D}$$

C

Apply Faraday's Law to the armature circuit and assume perfectly conducting wires.

$$\oint_C \overrightarrow{E} \cdot \overrightarrow{dl} = -\frac{d}{dt} \int_S \underbrace{\overrightarrow{B} \cdot \overrightarrow{dS}}_{zero} = 0$$

$$\underbrace{\int_{(+)}^{(-)}}_{fluid} E_y dy + \underbrace{\int_{(-)}^{(+)}}_{terminals} -\nabla\phi \cdot \overrightarrow{dl} = 0 \Rightarrow E_y W = v_A$$

Ohm's Law $\Rightarrow J = \sigma(E + v \times B) \Rightarrow E_y = \frac{J_y}{\sigma} + vB_z$

$$E_y = \frac{i_A}{\sigma DT} + vB_z$$

$$v_A = \underbrace{\left(\frac{W}{\sigma DT}\right)}_{R} i_A + \underbrace{\left(\frac{\mu_0 N W}{D}\right)}_{G} v i_F$$

D

Force density =

$$\overline{J} \times \overline{B} = J_y B_z \hat{x} = \frac{\mu_0 N i_F i_A}{TD} \hat{x}$$

Power =

$$J_y B_z U \cdot \underbrace{TDW}_{volume} = \frac{\mu_0 N W}{D} i_F i_A U = G i_F i_A U$$

E

$$v_A = R i_A + G U i_F$$
$$v_F = L \frac{di_F}{dt}$$
$$v_F = v_A$$
$$i_F = -i_A$$

Putting everything together,

$$L\frac{di_F}{dt} = -R i_F + G U i_F$$

Self excitation implies

$$GU > R \Rightarrow U > \frac{1}{\mu_0 \sigma N T}$$