6.641 Electromagnetic Fields, Forces, and Motion
Spring 2005

6.641 — Electromagnetic Fields, Forces, and Motion	Spring 2005
Problem Set 8 - Solutions	
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Problem 8.1

A

From Fig. 6P.1 we see the geometric relations

$$r' = r, \theta' = \theta - \Omega t, z' = z, t' = t$$

There is also a set of back transformations

$$r = r', \theta = \theta' + \Omega t, z = z', t = t' \quad (1)$$

B

Using the chain rule for partial derivatives

$$\frac{\partial \Psi}{\partial t'} = \left(\frac{\partial \Psi}{\partial r}\right)\left(\frac{\partial r}{\partial t'}\right) + \left(\frac{\partial \Psi}{\partial \theta}\right)\left(\frac{\partial \theta}{\partial t'}\right) + \left(\frac{\partial \Psi}{\partial z}\right)\left(\frac{\partial z}{\partial t'}\right) + \left(\frac{\partial \Psi}{\partial t}\right)\left(\frac{\partial t}{\partial t'}\right)$$

From (1) we learn that

$$\frac{\partial r}{\partial t'} = 0, \frac{\partial \theta}{\partial t'} = \Omega, \frac{\partial z}{\partial t'} = 0, \frac{\partial t}{\partial t'} = 1$$

Hence,

$$\frac{\partial \Psi}{\partial t'} = \frac{\partial \Psi}{\partial t} + \Omega \frac{\partial \Psi}{\partial \theta}$$

We note that the remaining partial derivatives of Ψ are

$$\frac{\partial \Psi}{\partial r'} = \frac{\partial \Psi}{\partial r}, \frac{\partial \Psi}{\partial \theta'} = \frac{\partial \Psi}{\partial \theta}, \frac{\partial \Psi}{\partial z'} = \frac{\partial \Psi}{\partial z}$$

Problem 8.2

A

In the frame rotating with the cylinder

$$\bar{E}'(r') = \frac{K}{r'}\bar{i}_r$$

$$\bar{H}' = 0, \bar{B}' = \mu_0 \bar{H}' = 0$$

But then since $r' = r, \bar{v}_r(r) = r\omega\bar{i}_\theta$

$$\bar{E} = \bar{E}' - \bar{v}_r \times \bar{B}' = \bar{E}' = \frac{K}{r}\bar{i}_r$$

$$V = \int_a^b \bar{E} \cdot d\bar{l} = \int_a^b \frac{K}{r} dr = K \ln(\frac{b}{a})$$

$$\bar{E} = \frac{V}{\ln\left(\frac{b}{a}\right)} \frac{1}{r} \overrightarrow{i_r} = \bar{E}' = \frac{V}{\ln\left(\frac{b}{a}\right)} \frac{1}{r'} \overrightarrow{i_r}$$

The surface charge density is then

$$\sigma'_a = \overrightarrow{i_r} \cdot \varepsilon_0 \bar{E}' = \frac{\varepsilon_0 V}{\ln \frac{b}{a}} \frac{1}{a} = \sigma_a$$

$$\sigma'_b = -\overrightarrow{i_r} \cdot \varepsilon_0 \bar{E}' = \frac{\varepsilon_0 V}{\ln \frac{b}{a}} \frac{1}{b} = \sigma_b$$

B

$$\bar{J} = \bar{J}' + \bar{v}_r \rho'$$

But in this problem we have only surface currents and charges

$$\bar{K} = \bar{K}' + \bar{v}_r \sigma' = \bar{v}_r \sigma'$$

$$\bar{K}(a) = \frac{a\omega\varepsilon_0 V}{a \ln\left(\frac{b}{a}\right)} \overrightarrow{i_\theta} = \frac{\omega\varepsilon_0 V}{\ln \frac{b}{a}} \overrightarrow{i_\theta}$$

$$\bar{K}(b) = -\frac{b\omega\varepsilon_0 V}{b \ln \frac{b}{a}} \overrightarrow{i_\theta} = -\frac{\omega\varepsilon_0 V}{\ln \frac{b}{a}} \overrightarrow{i_\theta}$$

C

$$\bar{H} = -\frac{\omega\varepsilon_0 V}{\ln \frac{b}{a}} \overrightarrow{i_z}$$

D

$$\bar{H} = \bar{H}' + \bar{v}_r \times \bar{D}' = \bar{v}_r \times \bar{D}'$$

$$\bar{H} = r'\omega \left(\frac{\varepsilon_0 V}{\ln \frac{b}{a}} \frac{1}{r'} \right) \left(\overrightarrow{i_\theta} \times \overrightarrow{i_r} \right)$$

$$\bar{H} = -\frac{\omega\varepsilon_0 V}{\ln \frac{b}{a}} \overrightarrow{i_z}$$

This result checks with the calculation of part (c).

Problem 8.3

A

We assume the simple magnetic field

$$\bar{H} = \begin{cases} -\frac{i}{D} \overrightarrow{i_3} & 0 < x_1 < x \\ 0 & x < x_1 \end{cases}$$

$$\lambda(x) = \int \bar{B} \cdot d\bar{a} = \frac{\mu_0 W x}{D} i$$

B

$$L(x) = \frac{\lambda(x,i)}{i} = \frac{\mu_0 W x}{D}$$

Since the system is linear

$$W'(i,x) = \frac{1}{2}L(x)i^2 = \frac{1}{2}\frac{\mu_0 W x}{D}i^2$$

C

$$f^e = \frac{\partial W'_m}{\partial x} = \frac{1}{2}\frac{\mu_0 W}{D}i^2 \tag{2}$$

D

The mechanical equation is

$$M\frac{d^2x}{dt^2} + B\frac{dx}{dt} = \frac{1}{2}\frac{\mu_0 W}{D}i^2 \tag{3}$$

The electrical circuit equation is

$$\frac{d\lambda}{dt} = \frac{d}{dt}\left(\frac{\mu_0 W x}{D}i\right) = V_0 \tag{4}$$

E

From (3) we learn that

$$\frac{dx}{dt} = \frac{\mu_0 W}{2BD}i^2 = \text{ constant}$$

while from (4) with i constant, we learn that

$$\frac{\mu_0 W i}{D}\frac{dx}{dt} = V_0$$

Solving these two simultaneously

$$\frac{dx}{dt} = \left[\frac{DV_0^2}{2\mu_0 W B}\right]^{\frac{1}{3}}$$

F

From (2)

$$i = \sqrt{\frac{2BD}{\mu_0 W}\frac{dx}{dt}} = \left(\frac{D}{\mu_0 W}\right)^{\frac{2}{3}}(2B)^{\frac{1}{3}}V_0^{\frac{1}{3}}$$

G

As in part A,

$$\bar{H} = \begin{cases} -\frac{i}{D}\overrightarrow{i_3} & 0 < x_1 < x \\ 0 & x < x_1 \end{cases}$$

H

The surface current $\bar{K}$ is

$$\bar{K} = -\frac{i(t)}{D}\bar{i}_2$$

The force on the short is

$$\bar{F} = \int \bar{J} \times \bar{B} dv = DW\bar{K} \times \left(\frac{\mu_0 \bar{H}_1 + \mu_0 \bar{H}_2}{2} \right) \tag{5}$$

$$= \frac{\mu_0 W}{2} i^2(t) \overrightarrow{i_1} \tag{6}$$

I

$$\nabla \times \bar{E} = \frac{\partial E_2}{\partial x_1}\overrightarrow{i_3} = -\frac{\partial \bar{B}}{\partial t} = \frac{\mu_0}{D}\frac{di}{dt}\overrightarrow{i_3}$$

$$\bar{E} = \left(\frac{\mu_0 x}{D}\frac{di}{dt} + C \right)\overrightarrow{i_2}$$

$$= \left(\frac{\mu_0 x}{D}\frac{di}{dt} - \frac{V(t)}{W} \right)\overrightarrow{i_2}$$

J

Choosing a contour with the right leg in the moving short, the left leg fixed at $x_1 = 0$

$$\oint_C \overrightarrow{E}' \cdot \overrightarrow{dl} = -\frac{d}{dt}\int_S \bar{B} \cdot d\bar{a}$$

Since $E' = 0$ in the short and we are only considering quasistatic fields

$$\oint \overrightarrow{E}' \cdot d\overrightarrow{l} = V(t) = Wx\mu_0 \frac{\partial H_0}{\partial t} + W\frac{dx}{dt}\mu_0 H_0$$

$$= \frac{d}{dt}\left(\frac{\mu_0 W x}{D} i(t) \right) \tag{7}$$

K

$$\bar{n} \times (\bar{E}^b) = V_n \bar{B}^b$$

Here

$$\bar{n} = \overrightarrow{i_1}, V_n = \frac{dx}{dt}, \bar{B}^b = -\frac{\mu_0 i}{D}\overrightarrow{i_3}$$

$$\bar{E}_b = \left(\frac{\mu_0 x}{D}\frac{di}{dt} - \frac{V(t)}{W} \right)\overrightarrow{i_2} = \left(-\frac{dx}{dt}\frac{\mu_0}{D} i \right)\overrightarrow{i_2}$$

$$V(t) = \frac{\mu_0 x W}{D}\frac{di}{dt} + \frac{\mu_0 i W}{D}\frac{dx}{dt} = \frac{d}{dt}\left(\frac{\mu_0 x W i}{D} \right) \tag{8}$$

L

Equations (6) and (2) are identical. Equations (8), (7), and (4) are identical if $V(t) = V_0$. Since we used (2) and (4) to solve the first part we would get the same answer using (6) and (7) in the second part.

M

Since $\frac{di}{dt} = 0$

$$\bar{E}_2(x) = -\frac{V(t)}{w}\overrightarrow{i_y} = -\frac{V_0}{W}\overrightarrow{i_y}$$

Problem 8.4

A

The electric field in the moving laminations is

$$\overrightarrow{E'} = \frac{\overrightarrow{J}'}{\sigma} = \frac{\overrightarrow{J}}{\sigma} = \frac{i}{\sigma A}\overrightarrow{i_z}$$

The electric field in the stationary frame is

$$\overrightarrow{E} = \overrightarrow{E}' - \bar{V} \times \bar{B} = \left(\frac{i}{\sigma A} + r\omega B_y\right)\overrightarrow{i_z}$$

$$B_y = -\frac{\mu_0 N i}{S}$$

$$V = \left(\frac{2D}{\sigma A} - \frac{\mu_0 2Dr\omega N}{S}\right) i$$

Now we have the $V - i$ characteristic of the device. The device is in series with an inductance and a load resistor $R_t = R_L + R_{\text{int}}$.

$$\left[R_t + \frac{2D}{\sigma A} - \frac{\mu_0 2DrN}{S}\omega\right] i + \frac{\mu_0 N^2 aD}{S}\frac{di}{dt} = 0$$

B

Let

$$R_1 = R_t + \frac{2D}{\sigma A} - \frac{2D\mu_0 rN\omega}{S}, L = \frac{\mu_0 N^2 aD}{S}$$

$$i = I_0 e^{-\frac{R_1}{L}t}$$

$$P_d = \frac{i^2}{R_L} = \frac{I_0^2}{R_L}\left[e^{-\frac{R_1}{L}t}\right]^2$$

If

$$R_1 = R_t + \frac{2D}{\sigma A} - \frac{2D\mu_0 rN\omega}{S} < 0$$

the power delivered is unbounded as $t \to \infty$.

C

As the current becomes large, the electrical nonlinearity of the magnetic circuit will limit the exponential growth and determine a level of stable steady state operation (see Fig. 6.4.12).

Problem 8.5

A

The armature circuit equation is

$$v_A = R_a i_a + G I_f \omega \tag{9}$$

The equation of motion is

$$J \frac{d\omega}{dt} = G I_f i_a$$

Which may be integrated to yield

$$\omega(t) = \frac{G}{J} \int_{-\infty}^{t} i_a(t) \tag{10}$$

Combining (10) with (9)

$$v_A = R_a i_a + \frac{(G I_f)^2}{J_r} \int_{-\infty}^{t} i_a(t)$$

We recognize that

$$C = \frac{J_r}{(G I_f)^2}$$

B

$$C = \frac{J_r}{(G I_f)^2} = \frac{(0.5)}{(1.5)^2(1)} = 0.22 \text{ farads}$$

Problem 8.6

$$(L_r + L_f)\frac{di_f}{dt} + i_f(R_r + R_f - G\omega) + \frac{1}{C}\int i_f dt = 0$$

$$\frac{d^2 i_f}{dt^2} + \frac{(R_r + R_f - G\omega)}{L_r + L_f}\frac{di_f}{dt} + \frac{1}{(L_r + L_f)C} i_f = 0$$

$$i_f = I e^{st}$$

$$s^2 + \frac{(R_r + R_f - G\omega)}{(L_r + L_f)} s + \frac{1}{(L_r + L_f)C} = 0$$

$$s = -\frac{R_r + R_f - G\omega}{2(L_r + L_f)} \pm \left[\left[\frac{R_r + R_f - G\omega}{2(L_r + L_f)}\right]^2 - \frac{1}{(L_r + L_f)C}\right]^{\frac{1}{2}}$$

A

Self excited if $-G\omega + R_r + R_f < 0 \Rightarrow \omega > \frac{R_r + R_f}{G}$

B

dc self-excitation

$$\left[\frac{R_r + R_f - G\omega}{2(L_r + L_f)}\right]^2 - \frac{1}{(L_r + L_f)C} > 0, C > \frac{4(L_r + L_f)}{[R_r + R_f - G\omega]^2}$$

ac self-excitation

$$\left[\frac{R_r + R_f - G\omega}{2(L_r + L_f)}\right]^2 - \frac{1}{(L_r + L_f)C} < 0 \Rightarrow C < \frac{4(L_r + L_f)}{[R_r + R_f - G\omega]^2}$$

C

Frequency $\omega_0 = \left[\frac{1}{(L_r+L_f)C} - \left[\frac{R_r+R_f-G\omega}{2(L_r+L_f)}\right]^2\right]^{\frac{1}{2}}$