6.641 Electromagnetic Fields, Forces, and Motion
Spring 2005

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Problem Set 1 - Solutions

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Problem 1.1

A

$$\overrightarrow{F} = q(\overrightarrow{E} + \overrightarrow{v} \times \overrightarrow{B}) \leftarrow \text{Lorentz Force Law}$$

In the steady state $\overrightarrow{F} = 0$, so

$$q\overrightarrow{E} = -q\overrightarrow{v} \times \overrightarrow{B} \Rightarrow \overrightarrow{E} = -\overrightarrow{v} \times \overrightarrow{B}$$

$$\overrightarrow{v} = \begin{cases} v_y \hat{i}_y & \text{pos. charge carriers} \\ -v_y \hat{i}_y & \text{neg. charge carriers} \end{cases}$$

$$\overrightarrow{B} = B_0 \hat{i}_z$$

so

$$\overrightarrow{E} = \begin{cases} -v_y B_0 \hat{i}_x & \text{pos. charge carriers} \\ v_y B_0 \hat{i}_x & \text{neg. charge carriers} \end{cases}$$

B

$$v_H = \Phi(x = d) - \Phi(x = 0) = -\int_0^d E_x dx = \int_d^0 E_x dx$$

$$v_H = \begin{cases} v_y B_0 d & \text{pos. charges} \\ -v_y B_0 d & \text{neg. charges} \end{cases}$$

C

As seen in part (b), positive and negative charge carriers give opposite polarity voltages, so answer is "yes."

Problem 1.2

By problem:

$$\rho = \begin{cases} \rho_b & r < b \\ \rho_a & b < r < a \end{cases}$$

Also, no σ_s at $r = b$, but nonzero σ_s at $r = a$ such that $\overrightarrow{E} = 0$ for $r > a$.

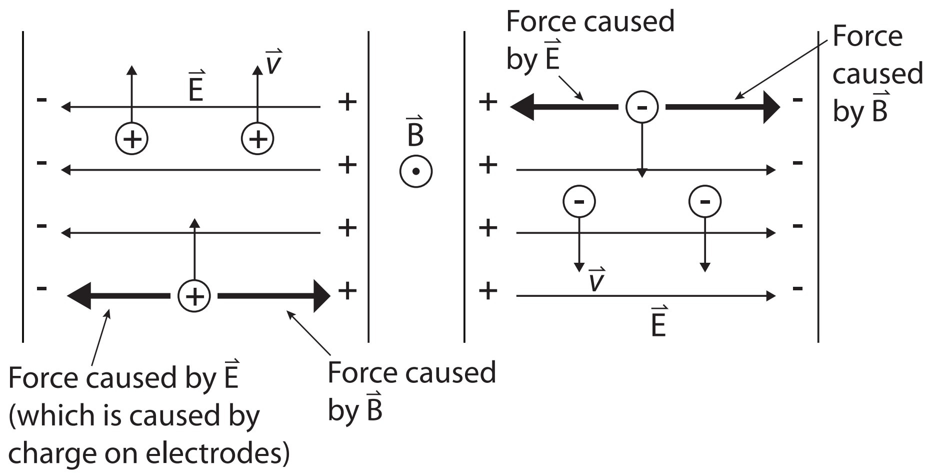

Figure 1: Figure for 1C. Opposite polarity voltages between holes and electrons (Image by MIT OpenCourseWare.)

A

By Gauss's Law

$$\oint_{S_R} \varepsilon_0 \overrightarrow{E} \cdot d\overrightarrow{a} = \int_{V_R} \rho dV \qquad S_R = \text{sphere with radius } r \tag{1}$$

As shown in class, symmetry ensures $\overrightarrow{E}$ has only radial component: $\overrightarrow{E} = E_r \hat{i}_r$.

LHS of (1):

$$\oint_{S_R} \varepsilon_0 \overrightarrow{E} \cdot d\overrightarrow{a} = \int_0^{2\pi} \int_0^{\pi} \epsilon_0 (E_r \hat{i}_r) \cdot \underbrace{r^2 \sin\theta d\theta d\phi \hat{i}_r}_{d\overrightarrow{a} \text{ in spherical coordinates}}$$

$$= \underbrace{4\pi r^2}_{\text{surface area of sphere of radius } r} E_r \varepsilon_0$$

RHS of (1):

For $r < b$:

$$\int_{V_R} \rho dV = \int_0^r \int_0^{2\pi} \int_0^{\pi} \rho_b \underbrace{r^2 \sin\theta d\theta d\phi dr}_{dV\text{: diff vol. element}}$$

$$= \underbrace{\frac{4}{3}\pi r^3}_{\text{vol of sphere}} \rho_b$$

For $r > b$ and $r < a$: $(b < r < a)$:

$$\begin{aligned}\int_{V_R} \rho dV &= \int_0^b \int_0^{2\pi} \int_0^{\pi} \rho_b r^2 \sin\theta d\theta d\phi dr \\ &+ \int_b^r \int_0^{2\pi} \int_0^{\pi} \rho_a r^2 \sin\theta d\theta d\phi dr \\ &= \frac{4\pi\rho_b b^3}{3} + \frac{4\pi\rho_a (r^3 - b^3)}{3}\end{aligned}$$

Equating LHS and RHS:

$$4\pi r^2 E_r \varepsilon_0 = \begin{cases} \frac{4\pi r^3}{3}\rho_b & r < b \\ \frac{4\pi \rho_b b^3}{3} + \frac{4\pi\rho_a(r^3-b^3)}{3} & b < r < a \end{cases}$$

$$E_r = \begin{cases} \frac{r\rho_b}{3\varepsilon_0} & r < b \\ \frac{b^3(\rho_b-\rho_a)}{3\varepsilon_0 r^2} + \frac{\rho_a r}{3\varepsilon_0} & b < r < a \end{cases}$$

B

Again: $\hat{n}\cdot(\varepsilon_0 E^a - \varepsilon_0 E^b) = \sigma_s$

$$\overrightarrow{E}(r = a^+) = 0$$

$$\overrightarrow{E}(r = a^-) = +\left[\frac{b^3(\rho_b-\rho_a)}{3\varepsilon_0 a^2} + \frac{\rho_a a}{3\varepsilon_0}\right]\hat{i}_r \qquad \text{by part (a)}$$

$$\sigma_s = \hat{i}_r \cdot (-\varepsilon_0 \overrightarrow{E}(r = a^-))$$

so

$$\sigma_s = -\left(\frac{b^3(\rho_b-\rho_a)}{3a^2} + \frac{\rho_a a}{3}\right)$$

C

$$r < b \qquad Q_b = \frac{4}{3}\pi b^3 \rho_b \qquad Q_\sigma(r = a) = \sigma_s 4\pi a^2$$

$$b < r < a \qquad Q_a = \frac{4}{3}\pi(a^3 - b^3)\rho_a$$

$$Q_T = Q_b + Q_a + Q_\sigma = 0$$

Problem 1.3

a

We are told current going in $+z$ direction inside cylinder $r < b$. Current going through cylinder

$$= I_{\text{total}} = \int_S \overrightarrow{J}\cdot d\overrightarrow{a}$$

$$= \int_0^b \int_0^{2\pi} \underbrace{(J_0\hat{i}_z)}_{\overrightarrow{J}} \cdot \underbrace{r d\phi dr \hat{i}_z)}_{d\overrightarrow{a}}$$

$$= J_0 \pi b^2$$

$$|\overrightarrow{K}| = \frac{\text{Total current in sheet}}{\text{length of sheet (i.e. circumference of circle of radius } a)}$$

Thus, $\overrightarrow{K}$'s units are $\frac{\text{Amps}}{m}$, whereas $\overrightarrow{J}$'s units are $\frac{\text{Amps}}{m^2}$

$$|\overrightarrow{K}| = \frac{J_0 \pi b^2}{2\pi a} = \frac{J_0 b^2}{2a}$$

$$\overrightarrow{K} = -\frac{J_0 b^2}{2a}\hat{i}_z$$

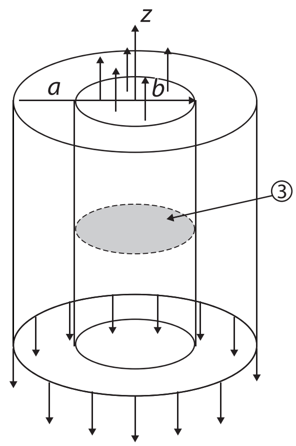

Figure 2: Figure for Problem 1.3 Part A. A cylinder with volume current going in the $+z$ direction for $r < b$. (Image by MIT OpenCourseWare)

B

$$\oint_C \overrightarrow{H} \cdot d\overrightarrow{s} = \int_S \overrightarrow{J} \cdot d\overrightarrow{a} + \underbrace{\frac{d}{dt}\int_S \varepsilon_0 \overrightarrow{E} \cdot d\overrightarrow{a}}_{\text{no } \overrightarrow{E} \text{ field, so this term is 0}} \qquad (2)$$

Choose C as a circle and S as the minimum surface that circle bounds. Now, solve LHS of Ampere's Law

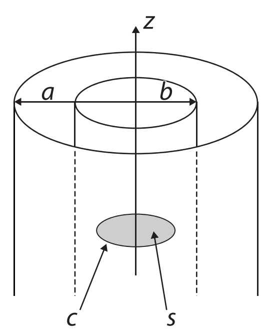

Figure 3: Choice of contour C and surface S (Image by MIT OpenCourseWare).

(2)

$$\oint_C \overrightarrow{H} \cdot d\overrightarrow{s} = \int_0^{2\pi} \underbrace{\left(H_\phi \hat{i}_\phi\right)}_{\overrightarrow{H}} \cdot \underbrace{\left(r d\phi \hat{i}_\phi\right)}_{d\overrightarrow{s}} = 2\pi r H_\phi$$

We assumed $H_z = H_r = 0$. This follows from the symmetry of the problem. $H_r = 0$ because $\oint_S \mu_0 \overrightarrow{H} \cdot d\overrightarrow{a} = 0$. In particular, choose S as shown in Figure 3. H_z is more difficult to see. It is discussed in Haus & Melcher. The basic idea is to use the contour, C, to show that if $H_z \neq 0$ it would have to be nonzero even at ∞, which is not possible without sources at ∞. Now for RHS of Ampere:

$r < b$:

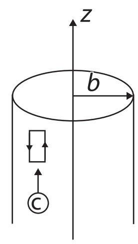

Figure 4: Illustration of the contour C (Image by MIT OpenCourseWare).

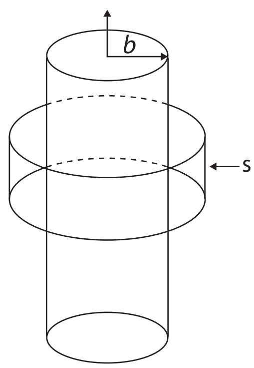

Figure 5: Diagram showing surface S (Image by MIT OpenCourseWare).

$$\int_S \overrightarrow{J} \cdot d\overrightarrow{a} = \int_0^{2\pi} \int_0^r \underbrace{\left(J_0 \cdot \hat{i}_z\right)}_{\overrightarrow{J}} \cdot \underbrace{\left(r^{'} dr^{'} d\phi \hat{i}_z\right)}_{d\overrightarrow{a}}$$

$$= J_0 r^2 \pi$$

$$a > r > b :$$

$$\int_S \overrightarrow{J} \cdot d\overrightarrow{a} = \int_0^{2\pi} \int_0^b (J_0 \hat{i}_z) \cdot (r^{'} dr^{'} d\phi \hat{i}_z + \underbrace{\int_0^{2\pi} \int_b^r (0 \cdot \hat{i}_z) \cdot (r^{'} dr^{'} d\phi \hat{i}_z)}_{0}$$

$$= J_0 b^2 \pi$$

Equating LHS and RHS:

$$2\pi r H_\phi = \begin{cases} J_0 r^2 \pi & r < b \\ J_0 b^2 \pi & a > r > b \end{cases}$$

$$\overrightarrow{H} = \begin{cases} \frac{J_0 r}{2} \hat{i}_\phi & r < b \\ \frac{J_0 b^2}{2r} \hat{i}_\phi & a > r > b \end{cases}$$

C

From text, Ampere's continuity condition:

$$\hat{n} \times (\overrightarrow{H}^a - \overrightarrow{H}^b) = \overrightarrow{K}$$

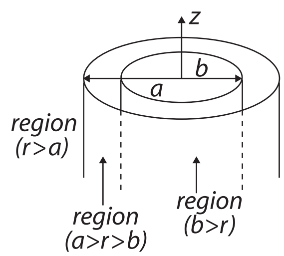

Figure 6: Ampere's continuity condition for Problem 1.3 Part C (Image by MIT OpenCourseWare)

$$H_\phi(r = a_-) = -K_z$$

$$\frac{J_0 b^2}{2a} = -K_z = \frac{J_0 b^2}{2a}$$

Problem 1.4

A

We can simply add the fields of the two point charges. Start with the field of a point charge q at origin and let S_R be sphere of radius R centered at the origin. By Gauss:

$$\oint_{S_R} \varepsilon_0 \overrightarrow{E} \cdot d\overrightarrow{a} = \int \rho dV$$

In this case $\rho = \delta(\overrightarrow{r})q$, so RHS is

$$\int \rho dV = \int\int\int \delta(\overrightarrow{r}) q dxdydz = q$$

LHS is

$$\oint_{S_R} \varepsilon_0 \overrightarrow{E_r} \cdot d\overrightarrow{a} = (\varepsilon_0 \underbrace{E_r}_{\text{symmetry again}})(\text{surface area of } S_r)$$

$$= 4\pi r^2 \varepsilon_0 E_r$$

Equate LHS and RHS

$$4\pi r^2 \varepsilon_0 E_r = q$$

$$\overrightarrow{E} = \frac{q}{4\pi r^2 \varepsilon_0} \hat{i}_r$$

Convert to cartesian: Any point is given by

$$\overrightarrow{r} = x(r,\theta,\phi)\hat{i}_x + y(r,\theta,\phi)\hat{i}_y + z(r,\theta,\phi)\hat{i}_z$$

By spherical coordinates

$$x = r\sin\theta\cos\phi$$

$$y = r\sin\theta\sin\phi$$

$z = r\cos\theta$
$\overrightarrow{r} = r\sin\theta\cos\phi\hat{i}_x + r\sin\theta\sin\phi\hat{i}_y + r\cos\theta\hat{i}_z$
$\hat{i}_r \parallel$ line formed by varying r and fixing ϕ and θ
$\bar{r} = r\bar{i_r}$

Thus,

$$\bar{i_r} = \sin\theta\cos\phi\hat{i_x} + \sin\theta\sin\phi\hat{i}_y + \cos\theta\hat{i}_z$$
$$= \frac{x}{\sqrt{x^2+y^2+z^2}}\hat{i_x} + \frac{y}{\sqrt{x^2+y^2+z^2}}\hat{i_y} + \frac{z}{\sqrt{x^2+y^2+z^2}}\hat{i_z}$$

so,

$$\overrightarrow{E} = \frac{q}{4\pi\varepsilon_0(x^2+y^2+z^2)}\hat{i}_r$$
$$\overrightarrow{E}_{\text{total}} = \overrightarrow{E}_1 + \overrightarrow{E}_2$$

$\overrightarrow{E}_1$ is just $\overrightarrow{E}$ with $y \to y - \frac{d}{2}$. $\overrightarrow{E}_2$ is just $\overrightarrow{E}$ with $y \to y + \frac{d}{2}$. Problem has $y = 0$

(i)

$$\overrightarrow{E}_{\text{total}} = \overrightarrow{E}_1 = \left[\frac{x}{\sqrt{x^2+\frac{d^2}{4}+z^2}}\hat{i}_x + \frac{z}{\sqrt{x^2+\frac{d^2}{4}+z^2}}\hat{i}_z - \frac{\frac{d}{2}}{\sqrt{x^2+\frac{d^2}{4}+z^2}}\hat{i}_y\right]\cdot\left[\frac{q}{4\pi\varepsilon_0(x^2+\frac{d^2}{4}+z^2)}\right]$$
$$= \frac{q}{4\pi\varepsilon_0(x^2+\frac{d^2}{4}+z^2)^{\frac{3}{2}}}\left[x\hat{i}_x - \frac{d}{2}\hat{i}_y + z\hat{i}_z\right]$$

(ii)

$$\overrightarrow{E}_{\text{total}} = \overrightarrow{E}_1 + \overrightarrow{E}_2$$
$$= q\frac{x\hat{i}_x + z\hat{i}_z}{2\pi\varepsilon_0(x^2+\left(\frac{d}{2}\right)^2+z^2)^{\frac{3}{2}}}$$

(iii)

$$\overrightarrow{E}_{\text{total}} = \overrightarrow{E}_1 + \overrightarrow{E}_2$$
$$= \frac{-dq\hat{i}_y}{4\pi\varepsilon_0\left(x^2+\frac{d^2}{4}+z^2\right)^{\frac{3}{2}}}$$

B

$\overrightarrow{F} = q_1\overrightarrow{E}$ $\qquad$ $\overrightarrow{E}$ doesn't include field of q

(i)

$\overrightarrow{F} = 0$, by Newton's third law a body cannot exert a net force on itself.

(ii)

$$\overrightarrow{F} = q_1\overrightarrow{E} = q\overrightarrow{E}_2(x=0, y=\frac{d}{2}, z=0)$$
$$= \frac{q^2\bar{i_y}}{4\pi\varepsilon_0(d^2)} = \frac{q^2\bar{i_y}}{4\pi\varepsilon_0 d^2}$$

(iii)

$$\overrightarrow{F} = -\frac{q^2\bar{i_y}}{4\pi\varepsilon_0 d^2}$$