Midterm solutions

Problem M.1 (60 points)

Your boss wants you to do a feasibility study for a digital communication system with the following characteristics.

You are allowed to use the frequency band B between 953 and 954 MHz. The allowed signal power is $P = 10^6$ power units. The noise in the band is additive white Gaussian noise with single-sided power spectral density $N_0 = 1$ power units per Hz.

For the purposes of the feasibility study, you may assume optimally bandwidth-efficient modulation, ideal brick-wall (zero-rolloff) filters, perfect receiver synchronization, etc.

(a) What is the Shannon limit on the achievable data rate R in bits per second (b/s)?

An AWGN channel is completely specified by its bandwidth W and signal-to-noise ratio SNR. The channel bandwidth is $W = |B| = 10^6$ Hz, and the SNR is $\mathrm{SNR} = P/(N_0 W) = 1$ (0 dB). The channel capacity in b/s is therefore

$$C_{\mathrm{b/s}} = W \log_2(1 + \mathrm{SNR}) = 10^6 \text{ b/s}.$$

We see that the nominal spectral efficiency is limited to $\rho < C_{\mathrm{b/2D}} = 1$ b/2D (or (b/s)/Hz), so we are in the power-limited regime.

(b) What is the maximum data rate R that can be achieved with uncoded modulation, if the target error rate is of the order of 10^{-5}*?*

In the power-limited regime, we use 2-PAM or (2×2)-QAM for uncoded modulation. From Figure 1, to achieve $P_b(E) \approx 10^{-5}$ requires $E_b/N_0 \approx 9.6$ dB, or $E_b/N_0 \approx 9$. Since $E_b = P/R$ and $P = 10^6$, $N_0 = 1$, this implies that

$$R \approx P/9 \approx 110,000 \text{ b/s},$$

a factor of about 9 (9.6 dB) less than capacity.

A number of you attacked this problem by varying ρ rather than R. Although this gives the right answer, it is not correct in principle because the nominal spectral efficiency of 2-PAM is a constant, $\rho = 2$ b/2D.

Note that the Nyquist (nominal) bandwidth at $R = 110,000$ b/s is only about 55 KHz; *i.e.,* with bandwidth-efficient modulation, only about 1/18 of the available channel bandwidth $W = 1$ MHz will be used.

(c) Suppose that for complexity reasons you are restricted to using Reed-Muller codes with block length $n \leq 64$. What is the maximum data rate R that can be achieved, if the target error rate is of the order of 10^{-5}?

From Table 2, we see that RM codes of length 64 can achieve up to about 6 dB of effective coding gain at $P_b(E) \approx 10^{-5}$. Thus to achieve $P_b(E) \approx 10^{-5}$ could require only $E_b/N_0 \approx 3.6$ dB, or $E_b/N_0 \approx 9/4 = 2.25$. Again using $E_b = P/R$ and $P = 10^6$, $N_0 = 1$, this implies that

$$R \approx 4P/9 \approx 440,000 \text{ b/s},$$

4 times the achievable rate of uncoded modulation. The required spectral efficiency at this data rate is $\rho \geq 0.44$ (b/s)/Hz.

Thus we can use the $(64, 22, 16)$ RM code, which has a nominal spectral efficiency of $\rho = 2k/n = 11/16 = 0.6875$ b/2D, and an effective coding gain of 6.0 dB. (We could not use the $(64, 7, 32)$ biorthogonal code, whose spectral efficiency is only $3/16 = 0.1875$.)

Using the $(64, 22, 16)$ code at a rate of $R \approx 440,000$ b/s, we will operate at $E_b/N_0 \approx 2.25$ (3.6 dB) and thus obtain $P_b(E) \approx 10^{-5}$, according to the union bound estimate. The nominal (Nyquist) bandwidth will be

$$W = R/\rho \approx \frac{16 \cdot 440,000}{11} = 640 \text{ KHz},$$

which is well within the 1 MHz available.

Now let the allowed signal power be only $P = 10^5$ power units, with all else the same.

(d) What is the Shannon limit on the achievable data rate R in bits per second (b/s)?

The channel bandwidth is still $W = 10^6$ Hz, but the SNR is now SNR $= 0.1$ (-10 dB). The channel capacity in b/s is therefore

$$C_{\text{b/s}} = W \log_2(1 + \text{SNR}) = W(\log_2 1.1) \approx (0.1375)W = 137,500 \text{ b/s}.$$

Since $\rho \leq 0.1375$ b/2D, we are now deep into the power-limited regime.

(e) What is the maximum data rate R that can be achieved with uncoded modulation, if the target error rate is of the order of 10^{-5}?

Again using 2-PAM or (2×2)-QAM, to achieve $P_b(E) \approx 10^{-5}$ requires $E_b/N_0 \approx 9.6$ dB, or $E_b/N_0 \approx 9$. Since $E_b = P/R$ and $P = 10^5$, $N_0 = 1$, this implies that

$$R \approx P/9 \approx 11,000 \text{ b/s},$$

a factor of about $9 \log_2 e$ (11.2 dB) less than capacity. Since we still have $\rho = 2$ b/2D, the Nyquist (nominal) bandwidth is now only about 5.5 KHz.

(f) Suppose that you are allowed to use any code that has been introduced in this course so far. What is the maximum data rate R that can be achieved, if the target error rate is of the order of 10^{-5}?

The best RM code in Table 2 is the $(64, 22, 16)$ code, which has an effective coding gain of 6.0 dB. Repeating the calculations in part (c) above for one tenth the power, we find that this code can support a rate of about

$$R \approx 4P/9 \approx 44,000 \text{ b/s},$$

4 times (6 dB) more than what we can achieve with uncoded modulation, but 3.3 times (5.2 dB) less than capacity.

The only potentially better codes that we know of so far are the orthogonal-simplex-biorthogonal family, whose performance we know approaches the ultimate Shannon limit on E_b/N_0 as $M \to \infty$, albeit with $\rho \to 0$.

Let us therefore try biorthogonal codes, the most bandwidth-efficient in this family. We know that the binary image of a $(2^m, m+1, 2^{m-1})$ binary linear block code is a 2^{m+1}-biorthogonal code.

If we are going to achieve a rate higher than 44 kb/s, then we are going to need an effective coding gain greater than 6 dB and a spectral efficiency greater than 0.044 b/2D.

The nominal coding gain of a $(2^m, m+1, 2^{m-1})$ code is $\gamma_c = kd/n = (m+1)/2$, so we are going to need to choose $m \geq 8$. The spectral efficiency is $\rho = 2k/n = (m+1)2^{-m+1}$, which limits us to $m \leq 8$, since for $m = 9$, $\rho = 10/256 = 0.039$ b/2D.

Let us therefore try the $(256, 9, 128)$ biorthogonal code. This code has a nominal coding gain of $\gamma_c = kd/n = 4.5$ (6.53 dB). It has $N_d = 2^9 - 2 = 510$ weight-128 codewords, so $K_b = N_d/k \approx 57$. Since this is about 6 factors of two, its effective coding gain is only about 5.3 dB, which is not good enough to improve on the $(64, 22, 16)$ code. So we conclude that no biorthogonal code can improve on the $(64, 22, 16)$ code in this problem.

Another approach is to examine RM codes of greater length. However, you were not expected to go further in this problem; consideration only of RM codes of lengths $n \leq 64$ and biorthogonal codes suffices for full credit.

[Extra credit] However, if you do go further, you will find for example that there is a $(128, 29, 32)$ RM code which can be constructed from the $(64, 22, 16)$ code and the $(64, 7, 32)$ code. This code has a nominal coding gain of $\gamma_c = kd/n = 29/4 = 7.25$ (8.6 dB), and a nominal spectral efficiency of $\rho = 2k/n = 29/64 = 0.45$, which are both fine. Its number of nearest neighbors is

$$N_d = 4\frac{127 \cdot 63 \cdot 31 \cdot 15 \cdot 7}{31 \cdot 15 \cdot 7 \cdot 3 \cdot 1} = 10668,$$

so $K_b = N_d/k \approx 368$, which is about 8.5 binary orders of magnitude. Its effective coding gain by our rule of thumb (which is questionable for such large numbers) is therefore about $\gamma_{\text{eff}} = 8.6 - 1.7 \approx 6.9$ dB. If this estimate is accurate, then the data rate can be improved by 0.9 dB, or about 23%; *i.e.,* to $R \approx 54,000$ b/s. However, the decoding complexity increases very significantly (trellis complexity $= 2^{15}$ states).

Grade distribution on Problem 1 ($N = 10$): $\{27, 28, 29, 34, 37, 41, 41, 47, 49, 52\}$.

Problem M.2 (40 points)

For each of the propositions below, state whether the proposition is true or false, and give a proof of not more than a few sentences, or a counterexample. No credit will be given for a correct answer without an adequate explanation.

(a) Let $\mathcal{A} = \{a_j(t), 1 \le j \le M\}$ be a set of M real $\mathcal{L}_2$ signals, and let the received signal be $r(t) = x(t) + n(t)$, where $x(t)$ is a signal in $\mathcal{A}$, and $n(t)$ is additive (independent) white Gaussian noise. Then, regardless of whether the signals in $\mathcal{A}$ are equiprobable or not, it is possible to do optimal detection on $r(t)$ by first computing from $r(t)$ a real M-tuple $\mathbf{r} = (r_1, r_2, \ldots, r_M)$, and then doing optimal detection on $\mathbf{r}$.

True. The signal space S spanned by $\mathcal{A}$ (*i.e.,* the set of all real linear combinations of signals in $\mathcal{A}$) is a real vector space with at most M dimensions (with equality if and only if the signals in $\mathcal{A}$ are linearly independent). It therefore has an orthonormal basis $\{\phi_j(t), 1 \le j \le \dim S\}$ consisting of $\dim S \le M$ orthonormal signals $\phi_j(t) \in \mathcal{L}_2$. By the theorem of irrelevance, the set of $\dim S \le M$ inner products $\{r_j = \langle r(t), \phi_j(t)\rangle, 1 \le j \le \dim S\}$ is a set of sufficient statistics for detection of signals in S in the presence of additive white Gaussian noise, regardless of their statistics.

It is also true that the set of M inner products $\{r'_j = \langle r(t), a_j(t)\rangle, 1 \le j \le M\}$ (the outputs of a bank of matched filters matched to each signal in $\mathcal{A}$) is another set of sufficient statistics for detection of signals in $\mathcal{A}$ in AWGN.

(b) Let $\mathcal{A}$ be an arbitrary M-point, N-dimensional signal constellation, and let $\mathcal{A}' = \alpha U \mathcal{A}^K$ be the constellation obtained by taking the K-fold Cartesian product $\mathcal{A}^K$, scaling by $\alpha > 0$, and applying an orthogonal transformation U. Then the effective coding gain of $\mathcal{A}'$ is the same as that of $\mathcal{A}$.

True. The effective coding gain at any target error rate is determined by the curve of $P_b(E)$ *vs.* E_b/N_0 in the power-limited regime, or of $P_s(E)$ *vs.* $\mathrm{SNR}_{\mathrm{norm}}$ in the bandwidth-limited regime. Both of these curves have been shown in homework problems to be invariant to scaling, orthogonal transformations, and the taking of Cartesian products.

(c) Let $\{\mathcal{C}_j, j = 1, 2, \ldots\}$ be an infinite set of binary linear (n_j, k_j, d_j) block codes $\mathcal{C}_j$ with $n_j \to \infty$ as $j \to \infty$. Then in order for the performance of these codes in AWGN to approach the Shannon limit as $j \to \infty$, it is necessary that either $\lim_{j\to\infty} k_j/n_j > 0$ or $\lim_{j\to\infty} d_j/n_j > 0$.

Most probably false. The nominal coding gain is $\gamma_c = kd/n$, and therefore we could have $\gamma_c \to \infty$ while both $k/n \to 0$ and $d/n \to 0$; *e.g.,* if $k \propto n^{2/3}$ and $d \propto n^{2/3}$. We know that biorthogonal codes approach the Shannon limit for $\rho \to 0$ with $k/n \to 0$ while $d/n \to \frac{1}{2}$, and we believe that rate-1/2 RM codes approach the Shannon limit for $\rho = 1$ with $d/n \to 0$ while $k/n \to \frac{1}{2}$. So there is every reason to believe that there should be sequences of intermediate codes that approach the Shannon limit for $\rho \to 0$ with $k/n \to 0$ and $d/n \to 0$. [Credit given for the quality of your discussion.]

(d) The Euclidean-space image $s(\mathcal{C})$ of a binary linear block code $\mathcal{C}$ under the 2-PAM map $\{s(0) = +\alpha, s(1) = -\alpha\}$ has zero mean, $\mathbf{m}(s(\mathcal{C})) = \mathbf{0}$, unless there is some coordinate position in which all codewords in $\mathcal{C}$ have value 0.

TRUE. This proposition holds if and only if half the codewords in any binary linear code $\mathcal{C}$ have a 0 in any coordinate position, and half have a 1 (unless all are 0).

To show this, take a given coordinate position, and assume that there is some codeword $\mathbf{x} \in \mathcal{C}$ that has a 1 in that coordinate position. Then we can partition the codewords $\mathcal{C}$ into $|\mathcal{C}|/2$ sets of pairs $(\mathbf{y}, \mathbf{x} + \mathbf{y}), \mathbf{y} \in \mathcal{C}$ (the cosets of the one-dimensional subcode $\mathcal{C}' = \{\mathbf{0}, \mathbf{x}\}$ in $\mathcal{C}$). If one member of a pair $(\mathbf{y}, \mathbf{x} + \mathbf{y})$ has a 1 in the given coordinate position, then the other has a 0, and *vice versa*. Hence precisely half the codewords in $\mathcal{C}$ have a 1 in any given coordinate position, unless none do. This argument holds for all coordinate positions.

GRADE DISTRIBUTION ON PROBLEM 2 ($N = 10$): $\{10, 14, 16, 28, 30, 30, 30, 30, 30, 32\}$.

GRADE DISTRIBUTION ON MIDTERM ($N = 10$): $\{43, 44, 57, 57, 60, 65, 71, 77, 79, 82\}$.