Final Exam

- You have 3 hours (9:00-12:00) to complete the test.
- This is a closed-book test, except that five $8.5'' \times 11''$ sheets of notes are allowed.
- Calculators are allowed (provided that erasable memory is cleared).
- There are three problems on the quiz. The first is a 7-part problem, each part worth 10 points. The second is a 5-part problem, each part worth 10 points. The third problem consists of 4 unrelated true-false questions, each worth 10 points.
- The problems are not necessarily in order of difficulty.
- A correct answer does not guarantee full credit and a wrong answer does not guarantee loss of credit. You should concisely indicate your reasoning and show all relevant work. The grade on each problem is based on our judgment of your level of understanding as reflected by what you have written.
- If we can't read it, we can't grade it.
- If you don't understand a problem, please ask.

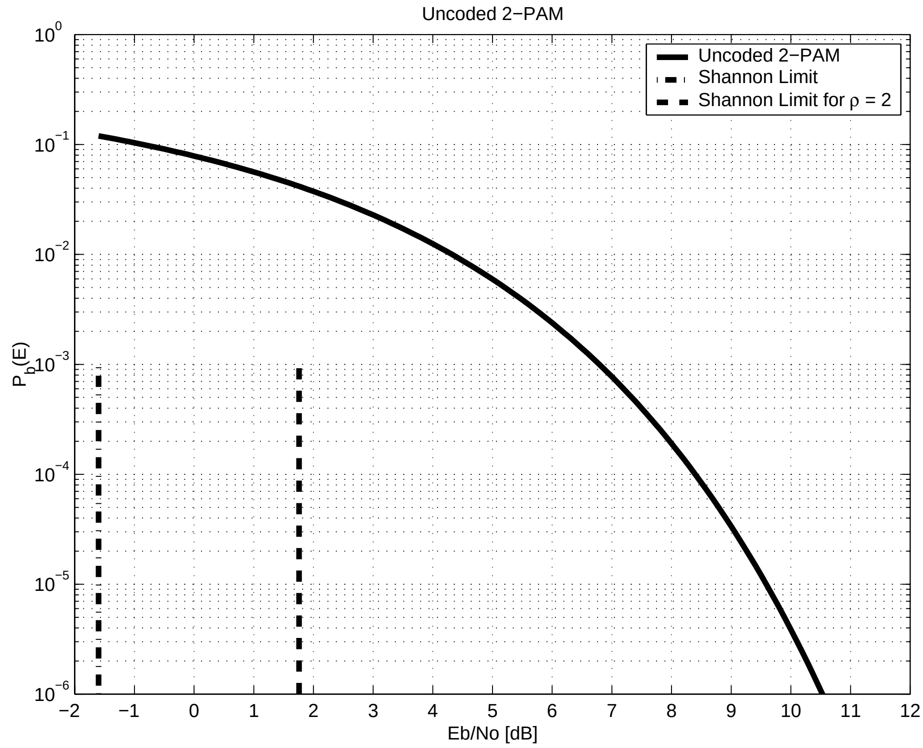

Figure 1. $P_b(E)$ *vs.* E_b/N_0 for uncoded binary PAM.

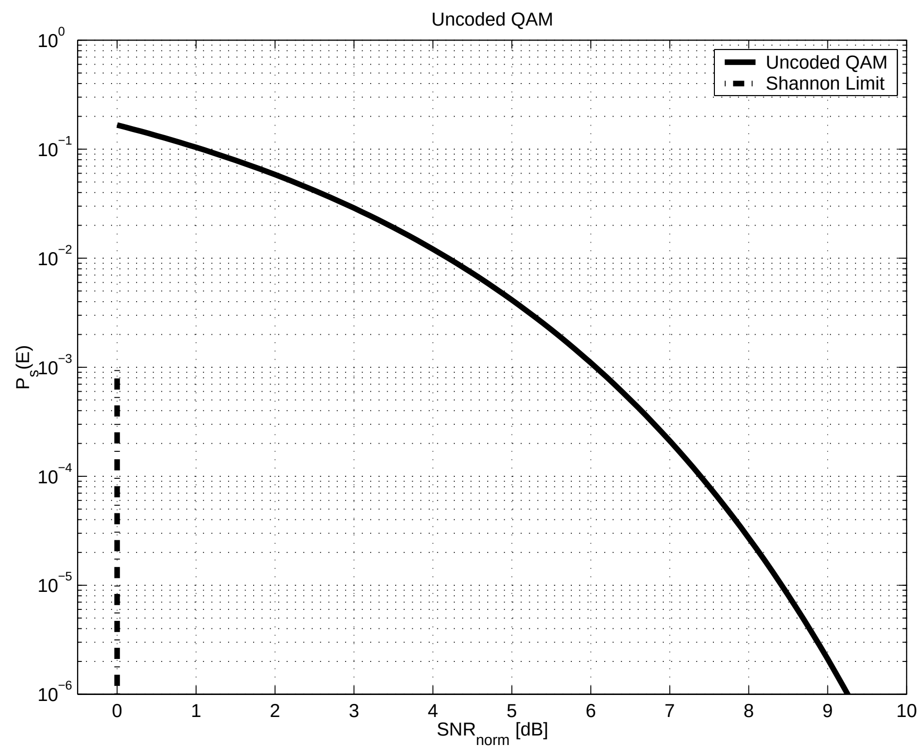

Figure 2. $P_s(E)$ *vs.* $\mathrm{SNR}_{\mathrm{norm}}$ for uncoded $(M \times M)$-QAM.

α	dB (round numbers)	dB (two decimal places)
1	0	0.00
1.25	1	0.97
2	3	3.01
2.5	4	3.98
e	4.3	4.34
3	4.8	4.77
π	5	4.97
4	6	6.02
5	7	6.99
8	9	9.03
10	10	10.00

Table 1. Values of certain small factors α in dB.

code	ρ	γ_c	(dB)	N_d	K_b	γ_{eff} (dB)	s	t
(8,7,2)	1.75	7/4	2.43	28	4	2.0	1	2
(8,4,4)	1.00	2	3.01	14	4	2.6	2	3
(16,15,2)	1.88	15/8	2.73	120	8	2.1	1	2
(16,11,4)	1.38	11/4	4.39	140	13	3.7	3	5
(16, 5,8)	0.63	5/2	3.98	30	6	3.5	3	4
(32,31, 2)	1.94	31/16	2.87	496	16	2.1	1	2
(32,26, 4)	1.63	13/4	5.12	1240	48	4.0	4	7
(32,16, 8)	1.00	4	6.02	620	39	4.9	6	9
(32, 6,16)	0.37	3	4.77	62	10	4.2	4	5
(64,63, 2)	1.97	63/32	2.94	2016	32	1.9	1	2
(64,57, 4)	1.78	57/16	5.52	10416	183	4.0	5	9
(64,42, 8)	1.31	21/4	7.20	11160	266	5.6	10	16
(64,22,16)	0.69	11/2	7.40	2604	118	6.0	10	14
(64, 7,32)	0.22	7/2	5.44	126	18	4.6	5	6

Table 2. Parameters of RM codes with lengths $n \leq 64$.

Problem F.1 (70 points)

In this problem we will consider coded modulation schemes based on a one-to-one mapping $t : \mathbb{F}_3 \to \mathcal{A}$ from the finite field $\mathbb{F}_3$ to a 3-simplex signal set $\mathcal{A}$ in $\mathbb{R}^2$ with energy $E(\mathcal{A})$ per symbol. The symbols from $\mathcal{A}$ will be transmitted by QAM modulation over a passband AWGN channel with single-sided power spectral density N_0. In everything that follows, we assume that the receiver performs optimal detection.

The amount of information that can be conveyed in one ternary symbol will be called *one trit*. We will normalize everything "per information trit;" *i.e.,*

- we will use E_t/N_0 as our normalized signal-to-noise ratio, where E_t is the average energy per information trit;
- we will define the nominal spectral efficiency ρ_t as the number of information trits per two dimensions (t/2D) conveyed by a given transmission scheme; and
- we will define $P_t(E)$ as the probability of error per information trit.

(a) What is the ultimate Shannon limit on E_t/N_0 in dB?

(b) What is the baseline performance ($P_t(E)$ *vs.* E_t/N_0) of the signal set $\mathcal{A}$?

(c) How far is this baseline performance from the ultimate Shannon limit at $P_t(E) \approx 10^{-5}$?

Let $\mathcal{C}$ be the $(4, 2, 3)$ linear "tetracode" over $\mathbb{F}_3$, and let $t(\mathcal{C})$ be the Euclidean image of $\mathcal{C}$ under the map $t : \mathbb{F}_3 \to \mathcal{A}$.

(d) What are the state and branch complexities of a minimal trellis for $\mathcal{C}$?

(e) What is the performance ($P_t(E)$ *vs.* E_t/N_0) of the signal set $t(\mathcal{C})$?

Now let $\mathcal{C}'$ be a linear rate-1/2 convolutional code over $\mathbb{F}_3$ with generator 2-tuple $\mathbf{g}(D) = (1 + D, 1 + 2D)$, and let $t(\mathcal{C}')$ be the Euclidean image of $\mathcal{C}'$ under the map t.

(f) What are the state and branch complexities of a minimal trellis for $\mathcal{C}'$?

(g) What is the performance ($P_t(E)$ *vs.* E_t/N_0) of $t(\mathcal{C}')$?

Problem F.2 (50 points)

Consider the $(16, 7, 6)$ binary linear block code $\mathcal{C}$ generated by the following generator matrix:

$$\begin{bmatrix} 1111 & 1100 & 0000 & 0000 \\ 0101 & 1011 & 1000 & 0000 \\ 1100 & 1001 & 0110 & 0000 \\ 1001 & 1111 & 0101 & 0000 \\ 1010 & 0001 & 0100 & 1100 \\ 1100 & 0101 & 0000 & 1010 \\ 0011 & 0010 & 0100 & 1001 \end{bmatrix}.$$

(a) It is known that $k_{\max}(n, 6) = \{0, 0, 0, 0, 0, 1, 1, 1, 2, 2, 3, 4, 4, 5, 6, 7\}$ for $1 \leq n \leq 16$. Show that there exist shortened codes of $\mathcal{C}$ that meet this bound for every $n \leq 16$.

(b) Give the state complexity profile and the branch complexity profile of a 16-section minimal trellis for $\mathcal{C}$.

(c) From the information given, is it possible to say whether another coordinate ordering might give a less complex trellis for $\mathcal{C}$?

(d) Find the sectionalization that gives the minimum number of sections without increasing the maximum branch complexity. Give the state complexity profile and the branch complexity profile of the resulting trellis.

(e) Count the number of arithmetic operations required by decoding using a straightforward Viterbi algorithm of the trellises of parts (b) and (d). Which is less complex?

Problem F.3 (40 points)

For each of the propositions below, state whether the proposition is true or false, and give a proof of not more than a few sentences, or a counterexample. No credit will be given for a correct answer without an adequate explanation.

(a) There exist sequences of Reed-Muller codes which can approach the Shannon limit arbitrarily closely, but the trellis complexity of such a sequence of codes necessarily grows without limit.

(b) Let G be a finite abelian group of order $|G|$, and let X and N be independent random variables defined on G, where the probability distribution of N is uniform:

$$p_N(n) = 1/|G|, \forall n \in G.$$

Then $Y = X + N$ is uniformly distributed over G and independent of X, regardless of the distribution of X.

(c) There exists no MDS binary linear block code with block length greater than 3.

(d) Given an (n, k, d) linear block code over a finite field $\mathbb{F}_q$ and optimal erasure correction:

(i) up to $d - 1$ erasures can always be corrected;

(ii) up to $n - k$ erasures may be able to be corrected;

(iii) more than $n - k$ erasures can never be corrected.