Quiz 2

- You have 90 minutes to complete the quiz.

- This is a closed-book quiz, except that hand written notes are allowed.

- Calculators are allowed, but will probably not be useful.

- There are three problems on the quiz.

- The problems are not necessarily in order of difficulty.

- A correct answer does not guarantee full credit and a wrong answer does not guarantee loss of credit. You should concisely indicate your reasoning and show all relevant work. The grade on each problem is based on our judgment of your level of understanding as reflected by what you have written.

- If we can't read it, we can't grade it.

- If you don't understand a problem, please ask.

Problem 1 (20 points) True or False: explain your answers, or better even, try to fix the false statements.

(a) Suppose $u(t)$ and $v(t)$ are both $\mathcal{L}_2$ and continuous. If $u(t)$ and $v(t)$ are $\mathcal{L}_2$ equivalent, then they are equal point-wise.

(b) If the Fourier transform $\hat{u}(f)$ of an $\mathcal{L}_2$ function $u(t)$ is zero for $|f| \geq W$, then $u(t)$ is completely described by its values at the sample points $u(jT)$ for all integer j and $T = 1/(2W)$.

(c) Suppose $u(t)$ is a real $\mathcal{L}_2$ waveform. Let $x(t) = u(t)\cos(2\pi f_c t)$, and $v(t) = x(t)\cos(2\pi f_c t)$. Then up to a scaling, $\hat{v}(f) = k \cdot \hat{u}(f)$ for some constant k and all $|f| \leq f_c$, in the $\mathcal{L}_2$ sense of equality.

(d) Let $x_1(t)$ and $x_2(t)$ be real $\mathcal{L}_2$ functions that are orthogonal to each other. Let $y_1(t)$ and $y_2(t)$ also be real orthogonal $\mathcal{L}_2$ functions. Then $x_1(t) + y_1(t)$ is orthogonal to $x_2(t) + y_2(t)$.

Problem 2 (30 points) Suppose $\hat{u}(f)$ is an L_2 function, and $\hat{u}(f) = 0, \forall |f| > W$. Let $u(t)$ be its inverse Fourier transform. We write $T = \frac{1}{2W}$. In class, we have shown that $u(t)$ can be reconstructed point-wise using its samples, $u(kT), k = 0, \pm 1, \pm 2, \ldots$, i.e.

$$u(t) = \sum_k u(kT)\text{sinc}\left(\frac{t}{T} - k\right), \qquad \forall t$$

(a) Suppose that $u(0) = 1$. However, for some reason this sample is lost and is replaced by $\tilde{u}(0) = 0$ to form the reconstruction

$$\tilde{u}(t) = \tilde{u}(0)\text{sinc}\left(\frac{t}{T}\right) + \sum_{k \neq 0} u(kT)\text{sinc}\left(\frac{t}{T} - k\right) \tag{1}$$

Compute the energy of the reconstruction error,

$$\int_{-\infty}^{\infty} |u(t) - \tilde{u}(t)|^2 dt$$

(b) Suppose that the sample $u(0)$ is not known, but it can be modelled as a random variable with a known distribution $P_{u_0}(\cdot)$. Choose a value of $\tilde{u}(0)$ such that the reconstruction (1) has the minimum average error energy. Explain.

(c) Recall that $\text{sinc}(t) \leftrightarrow \text{rect}(f)$ is a Fourier transform pair. Compute

$$\int_{-\infty}^{\infty} \text{sinc}(t) \cdot \text{sinc}\left(t - \frac{1}{2}\right) dt$$

(hint: little calculation is needed)

(d) Now suppose $u(0) = 1$, and that $u(kT) = 0, \forall k \neq 0$. However, we are only allowed to reconstruct $u(t)$ by

$$\tilde{u}(t) = \tilde{u}_1 \cdot \text{sinc}\left(\frac{t}{T} - \frac{1}{2}\right)$$

Find the best choice of $\tilde{u}_1$ that minimizes the energy of the reconstruction error, and compute the resulting error energy.

For the rest of the problem, you can assume $T = 1$.
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For the rest of the problem, you can assume $T = 1$.

(e) Following part (d), suppose now we are allowed to form

$$\tilde{u}(t) = \tilde{u}_1 \cdot \text{sinc}\left(\frac{t}{T} - \frac{1}{2}\right) + \tilde{u}_2 \cdot \text{sinc}\left(\frac{t}{T} + \frac{1}{2}\right)$$

Find the best choice of $\tilde{u}_1$ and $\tilde{u}_2$.

(f) Explain why using 2 sinc functions in part (e), the single missing sinc waveform still cannot be fully recovered. If one is allowed to use linear combinations of waveforms

$$\text{sinc}\left(\frac{t}{T} - m - \frac{1}{2}\right)$$

for any integer m, how many more waveforms are needed to drive the error energy to 0?

(g) Starting from part (d), suppose we no longer have the assumption that $u(kT) = 0, \forall k \neq 0$, but are allowed to form

$$\tilde{u}(t) = \tilde{u}_1 \cdot \text{sinc}\left(\frac{t}{T} - \frac{1}{2}\right) + \sum_{k \neq 0} u(kT)\text{sinc}\left(\frac{t}{T} - k\right)$$

Would your choice of $\tilde{u}_1$ be any different? Explain.

(h) Now suppose we know $u(kT)$ but are allowed to chose $\tilde{u}_k$ for all k (not necessarily the same as $u(kT)$) to form

$$\tilde{u}(t) = \tilde{u}_1 \cdot \text{sinc}\left(\frac{t}{T} - \frac{1}{2}\right) + \sum_{k \neq 0} \tilde{u}_k \cdot \text{sinc}\left(\frac{t}{T} - k\right)$$

Can we do any better than in (g)? Explain.