6.450 Principles of Digital Communications
MIT, Fall 2006

Wednesday October 11, 2005

Quiz 1

- You have 90 minutes to complete the quiz.
- This is a closed-book quiz, except that hand written notes are allowed.
- Calculators are allowed, but will probably not be useful.
- There are three problems on the quiz.
- The problems are not necessarily in order of difficulty.
- A correct answer does not guarantee full credit and a wrong answer does not guarantee loss of credit. You should concisely indicate your reasoning and show all relevant work. The grade on each problem is based on our judgment of your level of understanding as reflected by what you have written.
- If we can't read it, we can't grade it.
- If you don't understand a problem, please ask.

Problem Q1.1 Consider a random symbol X with the symbol alphabet $\{1, 2, \dots, M\}$ and a pmf $\{p_1, p_2, \dots, p_M\}$. This problem concerns the relationship between the entropy $H(X)$ and the probability p_1 of the first symbol. Let $Y = 1$ if $X = 1$ and $Y = 0$ otherwise. For parts (a) through (d), consider M and p_1 to be fixed.

(a) Express $H(Y)$ in terms of the binary entropy function, $H_b(\alpha) = -\alpha\log(\alpha) - (1-\alpha)\log(1-\alpha)$.

(b) What is the conditional entropy $H(X|Y = 1)$?

(c) Give a good upper bound to $H(X|Y = 0)$ and show how this bound can be met with equality by appropriate choice of $p_2, \dots, p_M$. Use this to upper bound $H(X|Y)$.

(d) Give a good upper bound for $H(X)$ and show how this bound can be met with equality by appropriate choice of $p_2, \dots, p_M$.

(e) For the same value of M as before, let $p_1, \dots, p_M$ be arbitrary and let $p_{\max}$ be $\max\{p_1, \dots, p_M\}$. Is your upper bound in (d) still valid if you replace p_1 by $p_{\max}$? Explain.

Problem Q1.2: Consider a DMS with i.i.d. $X_1, X_2, \ldots \in \mathcal{X} = \{a, b, c, d, e\}$, with probability $\{0.35, 0.25, 0.2, 0.1, 0.1\}$ respectively.

(a) Compute $\overline{L}_{\min}$, the expected codeword length of an optimal variable-length prefix free code for $\mathcal{X}$.

Let $\overline{L}_{\min}^{(2)}$ be the average codeword length, for an optimal code over $\mathcal{X}^2$, and $\overline{L}_{\min}^{(3)}$ as that for $\mathcal{X}^3$, and so on.

(c) True or False: for a general DMS, $\overline{L}_{\min} \geq \frac{1}{2}\overline{L}_{\min}^{(2)}$, explain.

(d) Show that $\overline{L}_{\min}^{(3)} \leq \overline{L}_{\min}^{(2)} + \overline{L}_{\min}$.

Problem Q1.3: In this problem, we try to construct a code which reduces the data rate at a cost of some amount of distortion in its reconstruction. Consider a binary source $X_1, X_2, \ldots$ i.i.d. Bernoulli (1/2) distributed. Obviously, a lossless source code would need 1 bit per source symbol to encode the source, allowing perfect reconstructions.

A lossy source code is defined as follows. An encoder map takes a source string X_1^n, encodes into nR bits, and a decoder reconstructs the source as $\hat{X}_1^n$. The goal is to guarantee that for any $\epsilon > 0$,

$$P_r\left(\frac{1}{n}|X_1^n - \hat{X}_1^n| > d + \epsilon\right) \to 0 \qquad \text{as } n \to \infty, \tag{1}$$

where $|X_1^n - \hat{X}_1^n|$ is the number of places that X_1^n and $\hat{X}_1^n$ are different.

The parameter d, which indicates the fraction of symbols that are allowed to be wrong, is often called a fidelity constraint. The lossless code we learned in class corresponds to the case that $d = 0$.

(a) Find the minimum rate of the lossy source code for the binary source above at $d = 1/2$, i.e., the reconstruction can have half of its symbols wrong in the sense of (1).

(b) To achieve $d = 1/4$, compare the following 2 approaches, both satisfying the fidelity constraint.

1) For a length $2n$ string, take the first n symbols and send uncoded, and ignore the rest. The decoder reconstruct the first n symbols, and simply lets $\hat{X}_{n+1}^{2n} = 0$.
2) For a length $2n$ string, divide it into 2 letter segments, which takes value 00, 01, 10, or 11. Construct a new binary string of length n, Z_1^n. Set $Z_i = 1$ if the i^{th} segment $X_{2i-1}^{2i} = 11$; and $Z_i = 0$ otherwise. Now the encoder applies a lossless code on Z, and transmits it. The decoder reconstructs Z, and for each Z_i, it reconstructs the i^{th} segment of $\hat{X}$. If $Z_i = 1$, the reconstruction $\hat{X}_{2i-1}^{2i} = 11$, otherwise $\hat{X}_{2i-1}^{2i} = 00$.

Compute the average rate of the two codes.

(c) (**bonus**) Do you think the better one of part (b) is optimal? If not, briefly explain your idea to improve over that.