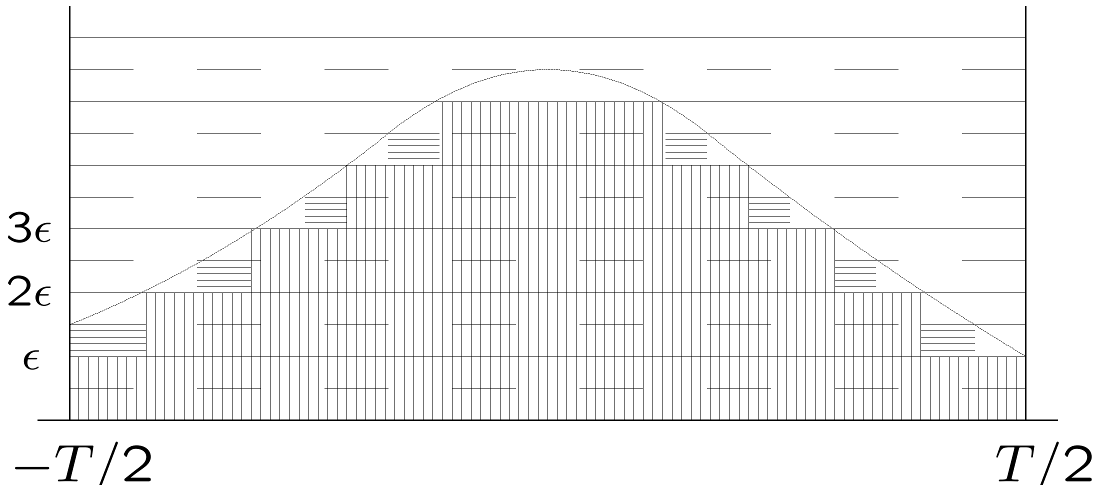
MEASURABLE FUNCTIONS

A function $\{u(t) : \mathbb{R} \to \mathbb{R}\}$ is measurable if $\{t : u(t) < b\}$ is measurable for each $b \in \mathbb{R}$.

The Lebesgue integral exists if the function is measurable and if the limit in the figure exists.



Horizontal crosshatching is what is added when $\varepsilon \to \varepsilon/2$. For $u(t) \geq 0$, the integral must exist (with perhaps an infinite value).

Theorem: Let $\{u(t) : [-T/2, T/2] \to \mathbb{C}\}$ **be an** $\mathcal{L}_2$ **function. Then for each** $k \in \mathbb{Z}$**, the Lebesgue integral**

$$\hat{u}_k = \frac{1}{T} \int_{-T/2}^{T/2} u(t)\, e^{-2\pi ikt/T}\, dt$$

exists and satisfies $|\hat{u}_k| \le \frac{1}{T} \int |u(t)|\, dt < \infty$**. Furthermore,**

$$\lim_{k_0 \to \infty} \int_{-T/2}^{T/2} \left| u(t) - \sum_{k=-k_0}^{k_0} \hat{u}_k\, e^{2\pi ikt/T} \right|^2 dt = 0,$$

where the limit is monotonic in k_0**.**

Given $\{\hat{u}_k; k \in \mathbb{Z}\}$**,** $\sum |\hat{u}_k|^2 < \infty$**, the** $\mathcal{L}_2$ **function** $u(t)$ **exists**

Functions not limited in time

We can segment an arbitrary $\mathcal{L}_2$ function into segments of any width T. The mth segment is $u_m(t) = u(t)\text{rect}(t/T - m)$. We then have

$$u(t) = \text{l.i.m.}_{m_0\to\infty} \sum_{m=-m_0}^{m_0} u_m(t)$$

This works because $u(t)$ is $\mathcal{L}_2$. The energy in $u_m(t)$ must go to 0 as $m \to \infty$.

$$u_m(t) = \text{l.i.m.} \sum_k \hat{u}_{k,m}\, e^{2\pi ikt/T}\, \text{rect}(\frac{t}{T} - m), \quad \textbf{where}$$

$$\hat{u}_{k,m} = \frac{1}{T}\int_{-\infty}^{\infty} u(t) e^{-2\pi ikt/T} \text{rect}(\frac{t}{T} - m)\, dt,$$

$$u(t) = \text{l.i.m.} \sum_{k,m} \hat{u}_{k,m}\, e^{2\pi ikt/T}\, \text{rect}(\frac{t}{T} - m)$$

Plancherel 1: There is an $\mathcal{L}_2$ function $\hat{u}(f)$ (the Fourier transform of $u(t)$), which satisfies the energy equation and

$$\lim_{A\to\infty} \int_{-\infty}^{\infty} |\hat{u}(f) - \hat{v}_A(f)|^2 \, dt = 0 \quad \textbf{where}$$

$$\hat{v}_A(f) = \int_{-A}^{A} u(t) e^{-2\pi i f t} \, dt.$$

We denote this function $\hat{u}(f)$ as

$$\hat{u}(f) = \text{l.i.m.} \int_{-\infty}^{\infty} u(t) e^{2\pi i f t} \, dt.$$

Although $\{\hat{v}_A(f)\}$ is continuous for all $A \in \mathbb{R}$, $\hat{u}(f)$ is not necessarily continuous.

Similarly, for $B > 0$, consider the finite bandwidth approximation $\hat{u}(f)\text{rect}(\frac{f}{2B})$. This is $\mathcal{L}_1$ as well as $\mathcal{L}_2$,

$$u_B(t) = \int_{-B}^{B} \hat{u}(f) e^{2\pi i f t} \, df \qquad (1)$$

exists for all $t \in \mathbb{R}$ and is continuous.

Plancherel 2: For any $\mathcal{L}_2$ function $u(t)$, let $\hat{u}(f)$ be the FT of Plancherel 1. Then

$$\lim_{B \to \infty} \int_{-\infty}^{\infty} |u(t) - w_B(t)|^2 \, dt = 0. \qquad (2)$$

$$u(t) = \text{l.i.m.} \int_{-\infty}^{\infty} \hat{u}(f) e^{2\pi i f t} \, df$$

<u>All</u> $\mathcal{L}_2$ functions have Fourier transforms in this sense.

The DTFT (Discrete-time Fourier transform) is the t↔f dual of the Fourier series.

Theorem (DTFT) Assume $\{\hat{u}(f) : [-W, W] \to \mathbb{C}\}$ **is** $\mathcal{L}_2$ **(and thus also** $\mathcal{L}_1$**). Then**

$$u_k = \frac{1}{2W} \int_{-W}^{W} \hat{u}(f) e^{2\pi i k f/(2W)} \, df$$

is a finite complex number for each $k \in \mathbb{Z}$**. Also**

$$\lim_{k_0 \to \infty} \int_{-W}^{W} \left| \hat{u}(f) - \sum_{k=-k_0}^{k_0} u_k \, e^{-2\pi i k f/(2W)} \right|^2 df = 0,$$

$$\hat{u}(f) = \text{l.i.m.} \sum_k u_k e^{-2\pi i f t/(2W)} \text{rect}\left(\frac{f}{2W}\right)$$

Sampling Theorem: Let $\{\hat{u}(f) : [-W\, W] \to \mathbb{C}\}$ **be** $\mathcal{L}_2$ **(and thus also** $\mathcal{L}_1$**). For** $u(t)$ **in (??), let** $T = 1/(2W)$**. Then the inverse transform** $u(t)$ **is continuous,** $\mathcal{L}_2$**, and bounded by** $u(t) \le \int_{-W}^{W} |\hat{u}(f)|\, df$**. For** $T = 1/(2W)$**,**

$$u(t) = \sum_{k=-\infty}^{\infty} u(kT)\ \text{sinc}\left(\frac{t - kT}{T}\right).$$

$$\hat{u}(f) = \sum_k u_k e^{-2\pi i k \frac{f}{2W}} \mathrm{rect}\left(\frac{f}{2W}\right)$$

$$u_k = \frac{1}{2W} \int_{-W}^{W} \hat{u}(f) e^{2\pi i k \frac{f}{2W}} \, df$$

Fourier series ⟷ **T/F dual** ⟷ **DTFT**

DTFT ⟷ **Fourier transform** ⟷ **Sampling**

$$u(t) = \sum_{k=-\infty}^{\infty} \hat{u}_k \, e^{2\pi i k t/T} \, \mathrm{rect}(\tfrac{t}{T})$$

$$\hat{u}_k = \frac{1}{T} \int_{-T/2}^{T/2} u(t) e^{-2\pi i k t/T} \, dt$$

$$u(t) = \sum_{k=-\infty}^{\infty} 2W u_k \, \mathrm{sinc}(2Wt - k)$$

$$u_k = \frac{1}{2W} u(\tfrac{k}{2W})$$

Segmenting an $\mathcal{L}_2$ frequency function into segments $\hat{v}_m(f) \longleftrightarrow v_m(t)$ of width $1/T$,

$$u(t) = \text{l.i.m.} \sum_{m,k} v_m(kT)\, \text{sinc}\left(\frac{t}{T} - k\right)\, e^{2\pi imt/T}.$$

Both this and the T-spaced truncated sinusoid expansion

$$u(t) = \text{l.i.m.} \sum_{m,k} \hat{u}_{k,m}\, e^{2\pi ikt/T} \text{rect}\left(\frac{t}{T} - m\right)$$

break the function into increments of time duration T and frequency duration $1/T$.

ALIASING

Suppose we approximate a function $u(t)$ that is not quite baseband limited by the sampling expansion $s(t) \approx u(t)$.

$$s(t) = \sum_k u(kT)\,\mathrm{sinc}\left(\frac{t}{T} - k\right).$$

$$u(t) = \text{l.i.m.} \sum_{m,k} v_m(kT)\,\mathrm{sinc}\left(\frac{t}{T} - k\right)\, e^{2\pi imt/T}$$

$$s(kT) = u(kT) = \sum_m v_m(kT) \qquad (\textbf{Aliasing})$$

$$s(t) = \sum_k \sum_m v_m(kT)\,\mathrm{sinc}\left(\frac{t}{T} - k\right).$$

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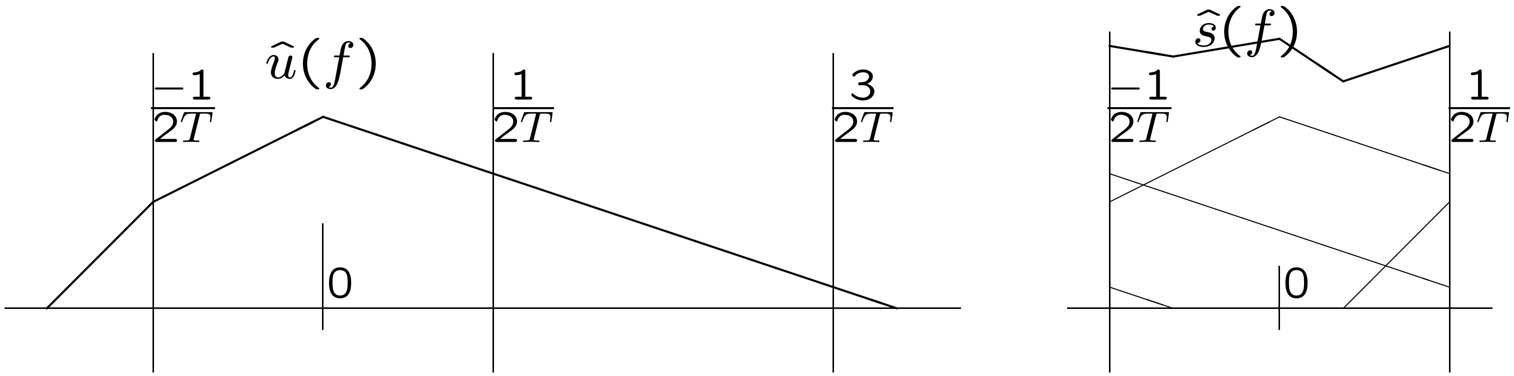


Theorem: Let $\hat{u}(f)$ be $\mathcal{L}_2$, and satisfy

$$\lim_{|f|\to\infty} \hat{u}(f)|f|^{1+\varepsilon} = 0 \qquad \textbf{for}\, \varepsilon > 0.$$

Then $\hat{u}(f)$ is $\mathcal{L}_1$, and the inverse transform $u(t)$ is continuous and bounded. For $T > 0$, the sampling approx. $s(t) = \sum_k u(kT)\,\mathrm{sinc}(\frac{t}{T} + k)$ is bounded and continuous. $\hat{s}(f)$ satisfies

$$\hat{s}(f) = \mathrm{l.i.m.} \sum_m \hat{u}(f + \frac{m}{T})\,\mathrm{rect}[fT].$$

$\mathcal{L}_2$ AS A VECTOR SPACE

Orthonormal expansions represent each $\mathcal{L}_2$ function as sequence of numbers.

View functions as vectors in inner product space, sequence as representation in a basis.

Same as $\mathbb{R}^k$ or $\mathbb{C}^k$ except for need of limiting operations.

The limits always exist for $\mathcal{L}_2$ functions in the sense of $\mathcal{L}_2$ convergence.

Any two functions that are equal except on a set of measure 0 are viewed as equal (same equivalence class).

Theorem: (1D Projection) Let $\mathbf{v}$ and $\mathbf{u} \neq 0$ be arbitrary vectors in a real or complex inner product space. Then there is a unique scalar α for which $\langle \mathbf{v} - \alpha\mathbf{u}, \mathbf{u}\rangle = 0$. That α is given by $\alpha = \langle \mathbf{v}, \mathbf{u}\rangle / \|\mathbf{u}\|^2$.

Proof: Calculate $\langle \mathbf{v} - \alpha\mathbf{u}, \mathbf{u}\rangle$ for an arbitrary scalar α and find the conditions under which it is zero:

$$\langle \mathbf{v} - \alpha\mathbf{u}, \mathbf{u}\rangle = \langle \mathbf{v}, \mathbf{u}\rangle - \alpha\langle \mathbf{u}, \mathbf{u}\rangle = \langle \mathbf{v}, \mathbf{u}\rangle - \alpha\|\mathbf{u}\|^2,$$

which is equal to zero if and only if $\alpha = \langle \mathbf{v}, \mathbf{u}\rangle / \|\mathbf{u}\|^2$.

Finite Projection: Assume that $\{\phi_1, \ldots, \phi_n\}$ is an orthonormal basis for an n-dimensional subspace $\mathcal{S} \subset \mathcal{V}$. For each $\mathbf{v} \in \mathcal{V}$, there is a unique $\mathbf{v}_{|\mathcal{S}} \in \mathcal{S}$ such that $\langle \mathbf{v} - \mathbf{v}_{|S}, \mathbf{s} \rangle = 0$ for all $\mathbf{s} \in \mathcal{S}$. Furthermore,

$$\mathbf{v}_{|\mathcal{S}} = \sum_j \langle \mathbf{v}, \phi_{\mathbf{j}} \rangle \, \phi_{\mathbf{j}}.$$

$$\|\mathbf{v}\|^2 = \|\mathbf{v}_{|\mathcal{S}}\|^2 + \|\mathbf{v}_{\perp\mathcal{S}}\|^2 \quad \textbf{(Pythagoras)}$$

$$0 \leq \|\mathbf{v}_{|\mathcal{S}}\|^2 \leq \|\mathbf{v}\|^2 \quad \textbf{(Norm bounds)}$$

$$0 \leq \sum_{j=1}^{n} |\langle \mathbf{v}, \phi_j \rangle|^2 \leq \|\mathbf{v}\|^2 \quad \textbf{(Bessel's inequality)}.$$

Gram-Schmidt: Given basis $\mathbf{s}_1, \dots, \mathbf{s}_n$ for an inner product subspace, find an orthonormal basis. Let $\phi_1 = \mathbf{s}_1/\|\mathbf{s}_1\|$. For each k,

$$\phi_{k+1} = \frac{(\mathbf{s}_{k+1})_{\perp \mathcal{S}_k}}{\|(s_{k+1})_{\perp \mathcal{S}_k}\|}$$

Infinite dimensional Projection theorem:

Let $\{\phi_m,\, 1 \le m < \infty\}$ be a set of orthonormal functions, and let $\mathbf{v}$ be any $\mathcal{L}_2$ vector. Then there is a unique $\mathcal{L}_2$ vector $\mathbf{u}$ such that $\mathbf{v} - \mathbf{u}$ is orthogonal to each ϕ_n and

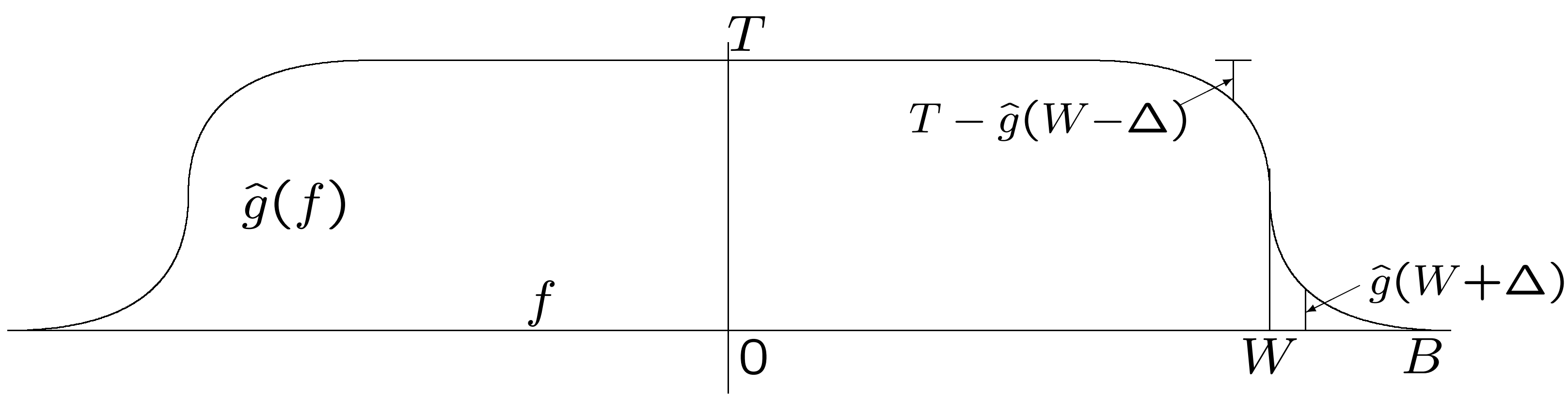
$$\lim_{n\to\infty} \|\mathbf{u} - \sum_{m=1}^{n} \langle \mathbf{v}, \phi_m \rangle \phi_m\| = 0.$$

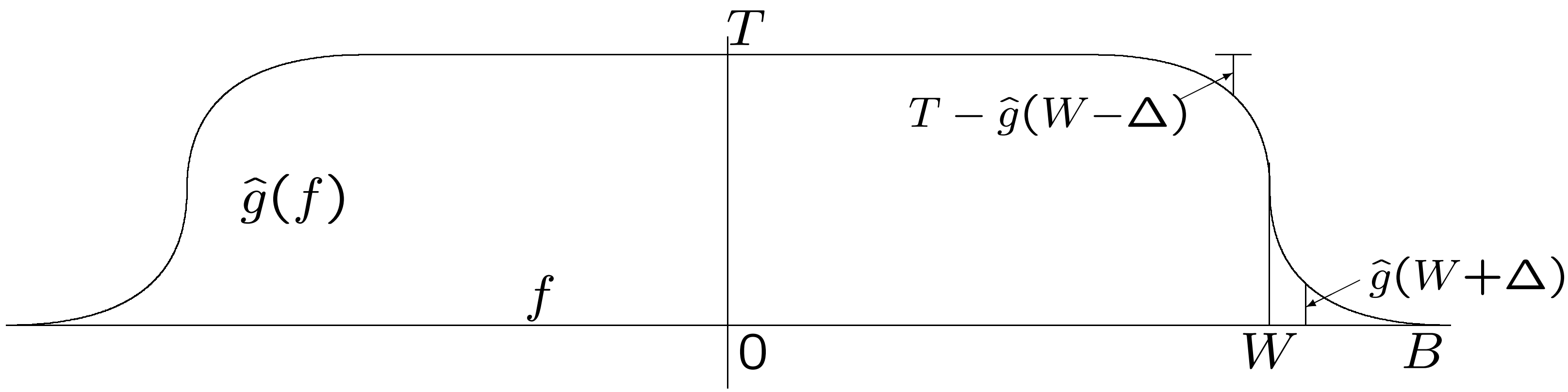
Nyquist: Convert $\{u_k\}$ into waveform $u(t) = \sum_k u_k p(t - kT)$.

At receiver, filter by $q(t)$, with combined effect $g(t) = p(t) * q(t)$.

Want $g(t)$ to be ideal Nyquist, i.e., $g(kT) = \delta_k$. $g(t)$ is ideal Nyquist iff

$$\sum_m \hat{g}(f + m/T)\,\mathrm{rect}(fT) = T\,\mathrm{rect}(fT)$$





Choose $\hat{g}(f)$ so that it cuts off quickly at W, but $g(t)$ cuts off relatively quickly at $1/T$.

Choose non-negative and symmetric (raised cosine for example)

Choose $q(t) = p^*(-t)$. Then $p(t)$ is orthogonal to its shifts.

A random process $\{Z(t)\}$ is a collection of rv's, one for each $t \in \mathbb{R}$.

For each epoch $t \in \mathbb{R}$, the rv $Z(t)$ is a function $Z(t, \omega)$ mapping sample points $\omega \in \Omega$ to real numbers.

For each $\omega \in \Omega$, $\{\mathbb{Z}(t, \omega\}$ is sample function $\{z(t)\}$.

A random process is defined by a rule establishing a joint density $f_{Z(t_1),\dots,Z(t_k)}(z_1, \dots, z_k)$ for all k, $t_1, \dots, t_k$ and $z_1, \dots, z_k$.

Our favorite way to do this is $\mathbf{Z}(t) = \sum Z_i \phi_i(t)$.

Joint densities on $Z_1, Z_2, \dots$ define $\{\mathbf{Z}(t)\}$.

A random vector $\mathbf{Z} = (Z_1, \dots, z_k)^\mathsf{T}$ of linearly independent rv's is jointly Gauss iff

1. $\mathbf{Z} = \mathbf{AN}$ **for normal rv N,**

2. $\mathbf{f_Z(z)} = \frac{1}{(2\pi)^{k/2}\sqrt{\det(\mathbf{K_Z})|}} \exp\left[-\frac{1}{2}\mathbf{z}^\mathsf{T}\mathbf{K_Z}^{-1}\mathbf{z}\right]$.

3. $\mathbf{f_Z(z)} = \prod_{j=1}^{k} \frac{1}{\sqrt{2\pi\lambda_j}} \exp\left[\frac{-|\langle \mathbf{z}, \mathbf{q}_j\rangle|^2}{2\lambda_j}\right]$ **for** $\{\mathbf{q}_j\}$ **orthonormal,** $\{\lambda_j\}$ **positive.**

4. **All linear combinations of Z are Gaussian.**

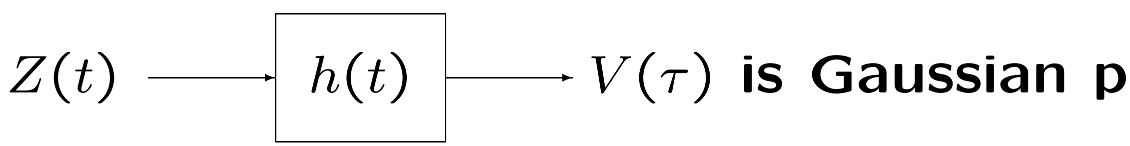
A linear functional of a rp is a rv given by

$$V = \int Z(t)g(t)\,dt.$$

This means that for all $\omega\in\Omega$**,**

$$V(\omega) = \langle Z(t,\omega), g(t)\rangle = \int_{-\infty}^{\infty} Z(t,\omega)g(t)\,dt.$$

If $Z(t) = \sum_j Z_j\phi_j(t)$ **is Gaussian process, then** $V = \sum_j Z_j\langle\phi_j, g\rangle$ **is Gaussian.**



$$\begin{aligned} V(\tau,\omega) &= \int_{-\infty}^{\infty} Z(t,\omega)h(\tau-t)\,dt \\ &= \sum_j Z_j(\omega)\int_{-\infty}^{\infty} \phi_j(t)h(\tau-t)\,dt. \end{aligned}$$

$\{Z(t); t \in \mathbb{R}\}$ **is stationary if** $Z(t_1), \dots, Z(t_k)$ **and** $Z(t_1+\tau), \dots, Z(t_k+\tau)$ **have same distribution for all** τ**, all** k**, and all** $t_1, \dots, t_k$**.**

Stationary implies that

$$\mathsf{K}_{\mathbf{Z}}(t_1, t_2) = \mathsf{K}_{\mathbf{Z}}(t_1 - t_2, 0) = \tilde{\mathsf{K}}_{\mathbf{Z}}(t_1 - t_2).$$

Note that $\tilde{\mathsf{K}}_{\mathbf{Z}}(t)$ **is real and symmetric.**

A process is wide sense stationary (WSS) if $\mathsf{E}[Z(t)] = \mathsf{E}[Z(0)]$ **and** $\mathsf{K}_{\mathbf{Z}}(t_1, t_2) = \mathsf{K}_{\mathbf{Z}}(t_1 - t_2, 0)$ **for all** t, t_1, t_2**.**

A Gaussian process is stationary if it is WSS.

An important example is $V(t) = \sum_k V_k \,\mathsf{sinc}(\frac{t-kT}{T})$**.**

If $\mathsf{E}[V_k V_i]{=}\sigma^2 \delta_{i,k}$**, then**

$$\mathsf{K}_{\mathbf{V}}(t,\tau) = \sigma^2 \sum_k \mathsf{sinc}\left(\frac{t-kT}{T}\right) \mathsf{sinc}\left(\frac{\tau-kT}{T}\right).$$

Then $\{V(t); t \in \mathbb{R}\}$ **is WSS with**

$$\tilde{\mathsf{K}}_{\mathbf{V}}(t-\tau) = \sigma^2 \,\mathsf{sinc}\left(\frac{t-\tau}{T}\right).$$

The sample functions of a WSS non-zero process are not $\mathcal{L}_2$.

The covariance $\tilde{\mathsf{K}}_{\mathbf{V}}(t)$ is $\mathcal{L}_2$ in cases of physical relevance. It has a Fourier transform called the <u>spectral density</u>.

$$S_{\mathbf{V}}(f) = \int \tilde{\mathsf{K}}_{\mathbf{V}}(t) e^{-2\pi i f t}\, dt$$

The spectral density is real and symmetric.

Let $V_j = \int Z(t) g_j(t)\, dt$. Then

$$\begin{aligned} \mathsf{E}[V_i V_j] &= \int_{t-\infty}^{\infty} g_i(t) \tilde{\mathsf{K}}_{\mathbf{Z}}(t-\tau) g_j(\tau)\, dt\, d\tau \\ &= \int \hat{g}_i(f) S_{\mathbf{Z}}(f) \hat{g}_j^*(f)\, df \end{aligned}$$

If $\hat{g}_i(f)$ and $\hat{g}_j(f)$ do not overlap in frequency, then $\mathsf{E}[V_i V_j] = 0$.

This means that for a WSS process, no linear functional in one frequency band is correlated with any linear functional in another band.

For a Gaussian stationary process, all linear functionals in one band are independent of all linear functionals in any other band; different frequency bands contain independent noise.

Summary of binary detection with vector observation in iid Gaussian noise.:

First remove center point from signal and its effect on observation.

Then signal is $\pm \mathsf{a}$. and $\mathsf{v} = \pm \mathsf{a} + \mathsf{Z}$.

Find $\langle \mathsf{v}, \mathsf{a} \rangle$ and compare with threshold (0 for ML case).

This does not depend on the vector basis - becomes trivial if a normalized is a basis vector.

Received components orthogonal to signal are irrelevant.

Review: Theorem of irrelevance

Given the signal set $\{\mathbf{a}_1, \dots, \mathbf{a}_M\}$, we transmit $X(t) = \sum_{j=1}^{k} a_{m,j}\phi_j(t)$ and receive $Y(t) = \sum_{j=1}^{\infty} Y_j\phi_j(t)$ where $Y_j = X_j + Z_j$ for $1 \le j \le k$ and $Y_j = Z_j$ for $j > k$.

Assume $\{Z_j;\, j \le k\}$ are iid and $\mathcal{N}(0, N_0/2)$. Assume $\{Z_j : j > k\}$ are arbitrary rv's that are independent of $\{X_j, Z_j;\, j \le k\}$.

Then the MAP detector depends only on $Y_1, \dots, Y_j$. The error probability depends only on $\{\mathbf{a}_1, \dots, \mathbf{a}_M\}$, and in fact, only on $\langle \mathbf{a}_j, \mathbf{a}_k \rangle$ for each $1 \le j, k \le M$.

Alll orthonormal expansions are the same; noise and signal outside of signal subspace can be ignored.

Orthogonal and simplex codes have the same error probability. The energy difference is $1-\frac{1}{m}$.

Orthogonal and biorthogonal codes have the same energy but differ by about 2 in error probability.

For orthogonal codes, take codewords as basis and normalize by $W_j = Y_j\sqrt{2/N_0}$. Thus the input for the first codeword is $(\alpha, 0, \ldots, 0)$ where $\alpha = \sqrt{2E/N_0}$. Then $W_j = \mathcal{N}(0,1)$ for $j \neq 0$ and $W_1 = a + \mathcal{N}(0,1)$.

$$\Pr(e) = \int_{-\infty}^{\infty} f_{W_1}(w_1) \Pr\left(\bigcup_{j=2}^{M} \{W_j \geq w_1\}\right) dw_1$$

Bottom line: Let $\log M = b$ **and** $E_b = E/b$**. Then**

$$\Pr(e) \leq \begin{cases} \exp\left[-b\left(\sqrt{\mathbf{E}_b/N_0} - \sqrt{\ln 2}\right)^2\right] & \textbf{for} \quad \frac{E_b}{4N_0} \leq \ln 2 < \frac{E_b}{N_0} \\ \exp\left[-b\left(\frac{\mathbf{E}_b}{2N_0} - \ln 2\right)\right] & \textbf{for} \quad \ln 2 < \frac{E_b}{4N_0} \end{cases}$$

This says we can get arbitrarily small error probability so long as $E_b/N_0 > \ln 2$**.**

This is Shannon's capacity formula for unlimited bandwidth WGN transmission.

Review of multipath model

The response to $\exp[2\pi i f t]$ **over** J **propagation paths with attenuation** β_j **and delay** $\tau_j(t)$ **is**

$$
\begin{aligned}
y_f(t) &= \sum_{j=1}^{J} \beta_j \exp[2\pi i f t - \tau_j(t)] \\
&= \hat{h}(f,t) \exp[2\pi i f t]
\end{aligned}
$$

The response to $x(t) = \int_{-\infty}^{\infty} \hat{x}(f) \exp[2\pi i f t]$ **is then**

$$
\begin{aligned}
y(t) &= \int_{-\infty}^{\infty} \hat{x}(f) \hat{h}(f,t) \exp(2\pi i f t)\, df \\
&= \int x(t-\tau)\, h(\tau,t)\, d\tau \qquad \textbf{where}
\end{aligned}
$$

$$
h(\tau,t) \longleftrightarrow \hat{h}(f,t); \qquad h(\tau,t) = \sum_j \beta_j \delta\{\tau - \tau_j(t)\}
$$

How do we define fading for a single frequency input?

$$
\begin{aligned}
y_f(t) &= \hat{h}(f,t)\,\mathsf{exp}[2\pi i f t] \\
&= |\hat{h}(f,t)|\,\mathsf{exp}[2\pi i f t + i\angle\hat{h}(f,t)] \\
\Re[y_f(t)] &= |\hat{h}(f,t)|\,\mathsf{cos}[2\pi f t + \angle\hat{h}(f,t)]
\end{aligned}
$$

The envelope of this is $|\hat{h}(f,t)|$, and this is defined as the fading.

$$\hat{h}(f,t) = \sum_j \beta_j\,\mathsf{exp}[-2\pi i f \tau_j(t)] = \sum_j \mathsf{exp}[2\pi i \mathcal{D}_j t - 2\pi i f \tau_j^o]$$

This contains frequencies ranging from $\min \mathcal{D}_j$ to $\max \mathcal{D}_j$. Define the Doppler spread of the channel as

$$\mathcal{D} = \max \mathcal{D}_j - \min \mathcal{D}_j$$

For any frequency Δ, $|\hat{h}(f,t)| = |e^{-2\pi i \Delta t}\hat{h}(f,t)|$

$$\hat{h}(f,t) = \sum_j \text{exp}\{2\pi i \mathcal{D}_j t - 2\pi i f \tau_j^o\}$$

Choose $\Delta = [\text{max}\, \mathcal{D}_j + \text{min}\, \mathcal{D}]/2$**. Then**

$$\text{exp}(-2\pi i t \Delta)\, \hat{h}(f,t) = \sum_{j=1}^{J} \beta_j \, \text{exp}\{2\pi i t(\mathcal{D}_j - \Delta) - 2\pi i f \tau_j^o\}$$

This waveform is baseband limited to $\mathcal{D}/2$**. Its magnitude is the fading. The fading process is the magnitude of a waveform baseband limited to** $\mathcal{D}/2$**. The coherence time of the channel is defined as**

$$\mathcal{T}_{\text{coh}} = \frac{1}{2\mathcal{D}}$$

$\mathcal{D}$ **is linear in** f**;** $\mathcal{T}_{\text{coh}}$ **goes as** $1/f$**.**

Review of time Spread

$$\hat{h}(f,t) = \sum_j \beta_j \exp[-2\pi i f \tau_j(t)]$$

For any given t, define

$$\mathcal{L} = \max \tau_j(t) - \min \tau_j(t); \qquad \mathcal{F}_{\text{coh}} = \frac{1}{2\mathcal{L}}$$

The fading at f is

$$|\hat{h}(f,t)| = \left| \sum_j \exp[2\pi i(\tau_j(t) - \tau')f] \right| \qquad \textbf{(ind. of } \tau'\textbf{)}$$

Let $\tau' = \tau_{\text{mid}} = (\max \tau_j(t) + \min \tau_j(t))/2$. The fading is the magnitude of a function of f with transform limited to $\mathcal{L}/2$. $\mathcal{T}_{\text{coh}}$ is a gross estimate of the frequency over which the fading changes significantly.

Baseband system functions

The baseband response to a complex baseband input $u(t)$ is

$$v(t) = \int_{-W/2}^{W/2} \hat{u}(f)\hat{h}(f+f_c,t)\, e^{2\pi i(f-\Delta)t}\, df$$
$$= \int_{-W/2}^{W/2} \hat{u}(f)\, \hat{g}(f,t)\, e^{2\pi i f t}\, df$$

where $\hat{g}(f,t) = \hat{h}(f+f_c,t)e^{-2\pi i \Delta t}$ is the baseband system function and $\Delta = \tilde{f}_c - f_c$ is the frequency offset in demodulation.

By the same relationship between frequency and time we used for bandpass,

$$v(t) = \int_{-\infty}^{\infty} u(t-\tau)g(\tau,t)\, d\tau$$

$$
\begin{aligned}
\widehat{h}(f,t) &= \sum_j \beta_j \,\mathsf{exp}\{-2\pi i f \tau_j(t)\} \\
\widehat{g}(f,t) &= \sum_j \beta_j \,\mathsf{exp}\{-2\pi i (f+f_c)\tau_j(t) - 2\pi i \Delta t\} \\
\widehat{g}(f,t) &= \sum_j \gamma_j(t) \,\mathsf{exp}\{-2\pi i f \tau_j(t)\} \qquad \textbf{where} \\
\gamma_j(t) &= \beta_j \,\mathsf{exp}\{-2\pi i f_c \tau_j(t) - 2\pi i \Delta t\} \\
&= \beta_j \,\mathsf{exp}\{2\pi i [\mathcal{D}_j - \Delta] t - 2\pi i f_c \tau_j^o \\
g(\tau,t) &= \sum_j \gamma_j(t)\delta(\tau - \tau_j(t)) \\
v(t) &= \sum_j \gamma_j(t) u(t - \tau_j(t))
\end{aligned}
$$

Flat fading

Flat fading is defined as fading where the bandwidth $W/2$ of $u(t)$ is much smaller than $\mathcal{F}_{\text{coh}}$.

For $|f| < W/2 << \mathcal{F}_{\text{coh}}$,

$$\hat{g}(f,t) = \sum_j \gamma_j(t)\,\text{exp}\{-2\pi i f \tau_j(t)\} \approx \hat{g}(0,t) = \sum_j \gamma_j(t)$$

$$v(t) = \int_{-W/2}^{W/2} \hat{u}(f)\,\hat{g}(f,t)\,e^{2\pi i f t}\,df \approx u(t)\sum_j \gamma_j(t)$$

Equivalently, $u(t)$ is approximately constant over intervals much less than $\mathcal{L}$.

$$v(t) = \sum_j \gamma_j(t) u(t - \tau_j(t)) = u(t)\sum_j \gamma_j(t)$$