Review: Theorem of irrelevance

Given the signal set $\{\vec{a}_1, \dots, \vec{a}_M\}$, we transmit $X(t) = \sum_{j=1}^{k} a_{m,j}\phi_j(t)$ and receive $Y(t) = \sum_{j=1}^{\infty} Y_j\phi_j(t)$ where $Y_j = X_j + Z_j$ for $1 \le j \le k$ and $Y_j = Z_j$ for $j > k$.

Assume $\{Z_j;\, j \le k\}$ are iid and $\mathcal{N}(0, N_0/2)$. Assume $\{Z_j : j > k\}$ are arbitrary rv's that are independent of $\{X_j, Z_j;\, j \le k\}$.

Then the MAP detector depends only on $Y_1, \dots, Y_j$. The error probability depends only on $\{\vec{a}_1, \dots, \vec{a}_M\}$, and in fact, only on $\langle \vec{a}_j, \vec{a}_k \rangle$ for each j, k.

Alll orthonormal expansions are the same; noise and signal outside of signal subspace can be ignored.

Next let $X(t) = \sum_n X_n(t)$ where $X_n(t) = \sum_j a_{m,j}^{(n)} \phi_j^{(n)}(t)$ is the nth of a sequence of modulated waveforms and $\phi_j^{(n)}(t)$ are orthonormal over j and n.

If the choice of $X_n(t)$ (over signals $\vec{a}_m$) is statistically independent from one n to another, then the optimal sequence detector is simply the optimal detector for one signal at a time.

With statistical dependence between $X_n(t)$, then the error probability for optimal sequence detection is less than or equal to that for successive independent detection.

This is true both for single-signal error probability and block error probability.

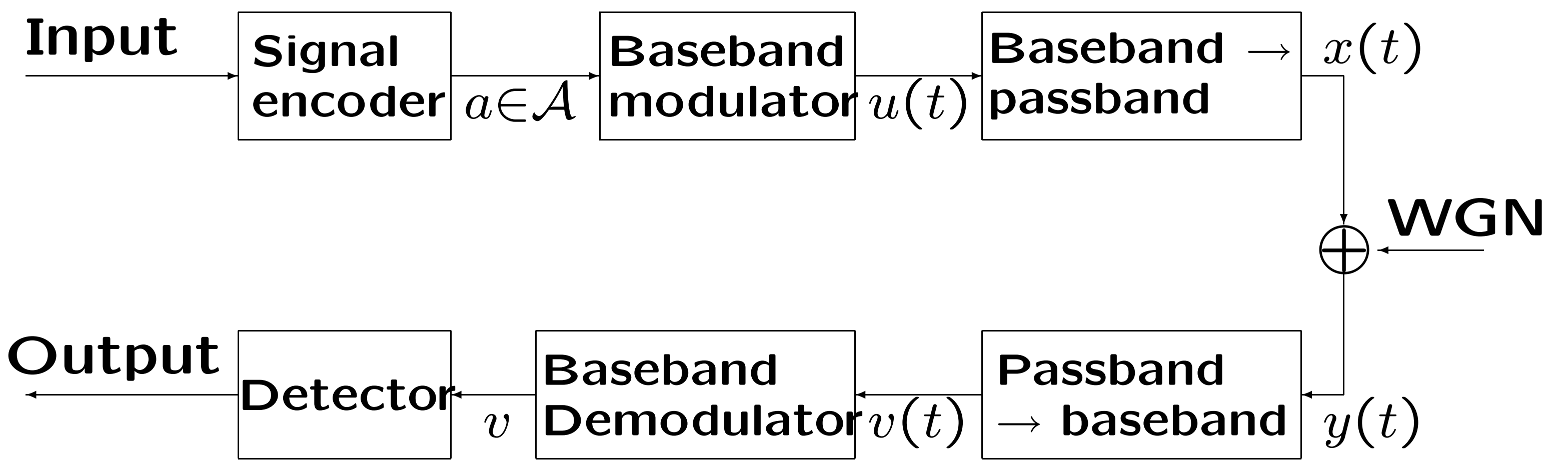
If $\{\phi_j(t); j \in \mathbb{Z}\}$ **is an orthonormal complex set at baseband, then**

$$\Psi_{j1}(t) = \Re\{2\phi_j(t)e^{2\pi i f_c t}\}; \qquad \Psi_{j2}(t) = \Im\{2\phi_j(t)e^{2\pi i f_c t}\}$$

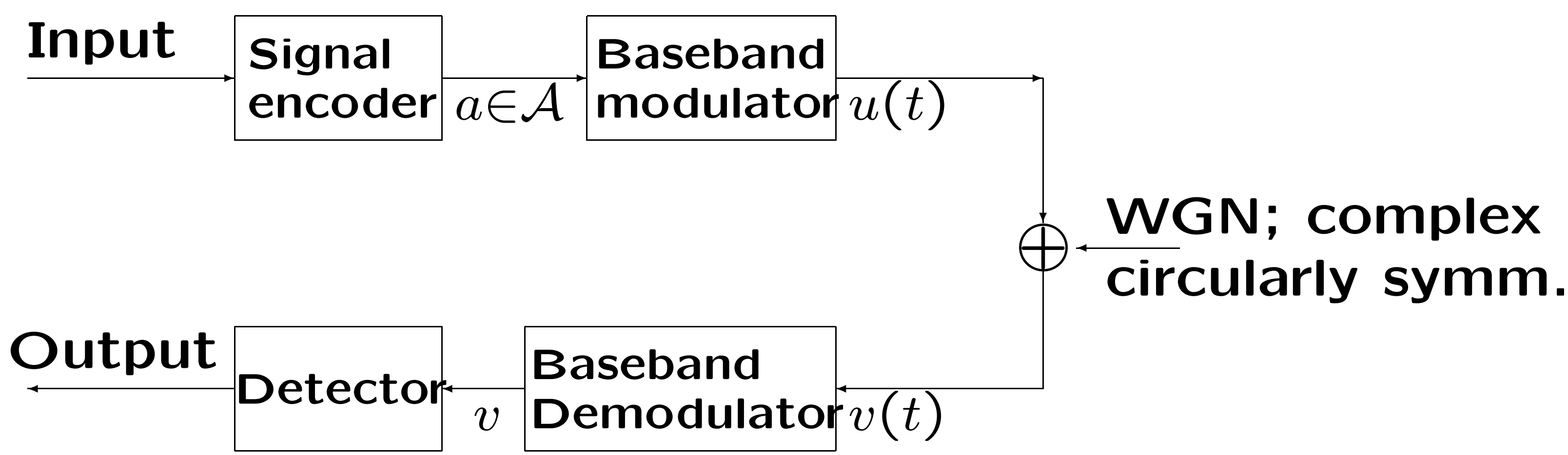
$$\begin{aligned} u(t) &= \sum_j a_j\phi_j(t) \;\to\; x(t) = \sum_j a_{j1}\Psi_{j1}(t) + a_{j2}\Psi_{j2}(t) \\ &\to\; y(t) = \sum_j (a_{j1} + Z_{j1})\Psi_{j1}(t) + (a_{j2} + Z_{j2})\Psi_{j2}(t) \\ &\to\; v(t) = \sum_j (a_j + Z_j) \sum_j a_j\phi_j(t) \end{aligned}$$

Here $\{Z_j; j \in \mathbb{Z}\}$ **is a sequence of iid circularly symmetric complex Gaussian rv's.**

Under complex linear transformations, the resulting noise rv's are Gaussian circularly symmetric.



Equivalent system

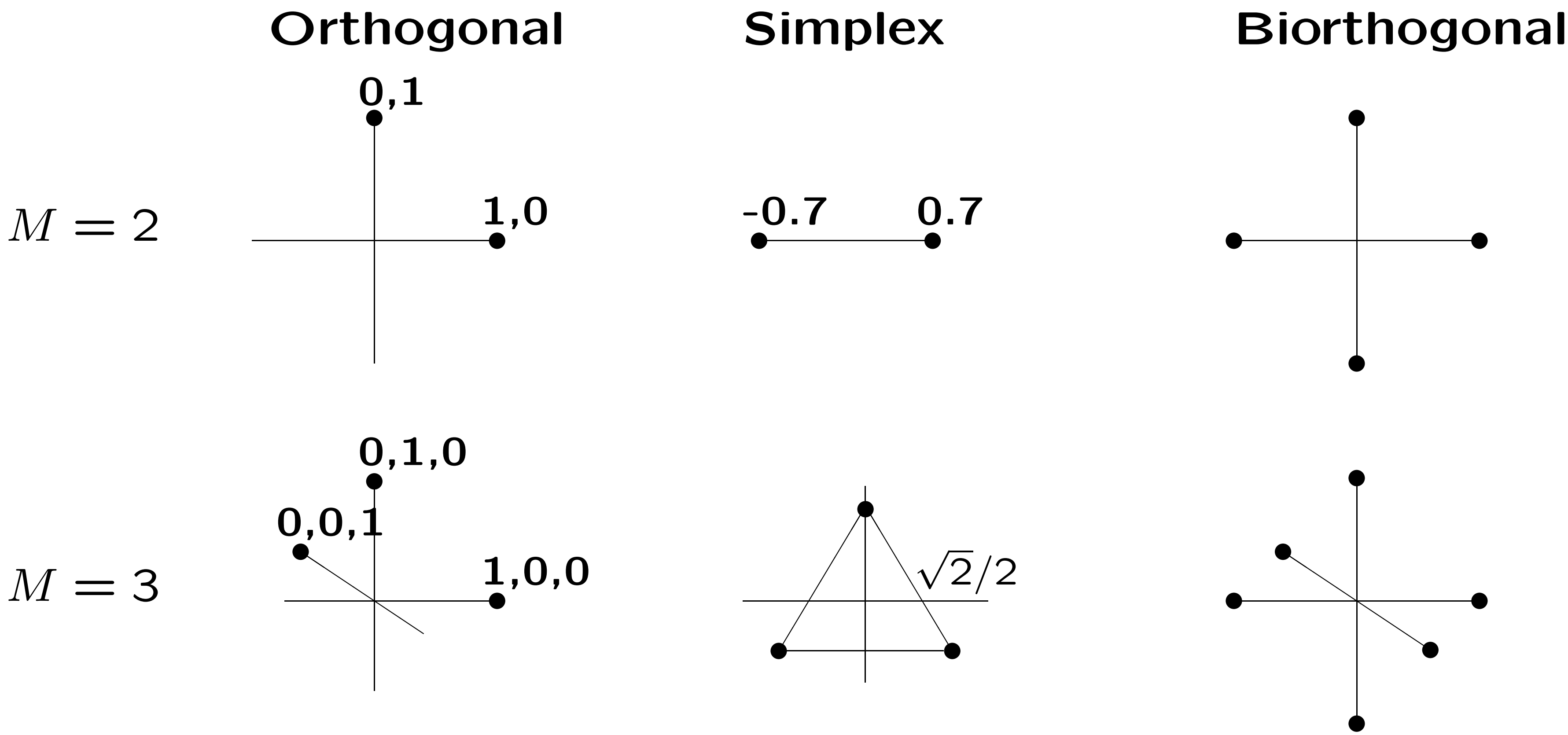


A set of signals $\vec{a}_1, \ldots, \vec{a}_M$ are orthogonal if $\langle \vec{a}_i, \vec{a}_j \rangle = E\delta_{ij}$ for $1 \leq i, j \leq M$. They span an M dimensional space and can be taken as basis vectors in $\mathbb{R}^M$.

The mean of an orthogonal set is $\vec{A} = (\frac{\sqrt{E}}{M}, \ldots, \frac{\sqrt{E}}{M})^\mathsf{T}$

The set $\vec{s}_j = \vec{a}_j - \vec{A}$ is a simplex code. This spans an $M - 1$ dimensional space. The energy is $E\frac{M-1}{M}$.

The set $\pm\vec{a}_1, \pm\vec{a}_2, \ldots, \pm\vec{a}_M$ is a biorthogonal code.



Note that for $M \geq 3$, the lines connecting closest points are not orthogonal.

Orthogonal and simplex codes have the same error probability. The energy difference is $1-\frac{1}{m}$.

Orthogonal and biorthogonal codes have the same energy but differ by about 2 in error probability.

We find the ML error probability for orthogonal codes. By symmetry, doesn't depend on codeword (signal), so assume input 1.

Normalize the output by $W_j = Y_j\sqrt{2/N_0}$. Thus the input is $(\alpha, 0, \ldots, 0)$ where $\alpha = \sqrt{2E/N_0}$.

Given this input, $W_1 \sim \mathcal{N}(\alpha, 1)$, $W_j \sim \mathcal{N}(0, 1)$ for $j \geq 2$ and $W_1, \ldots, W_M$ are independent.

An error is made is $W_j \geq W_1$ for any $2 \leq j \leq M$.

$$\Pr(e) = \int_{-\infty}^{\infty} f_{W_1}(w_1) \Pr\left(\bigcup_{j=2}^{M} \{W_j \geq w_1\}\right) dw_1$$

If w_1 is very small, then lots of other signals look more likely; if large, then union bound is good.

Let $B_1, B_2, \ldots B_n$ be independent equiprobable events of probability p.

$$\begin{aligned}
\Pr(\bigcup_{j=1}^{n} B_j) &= 1-(1-p)^n \leq \begin{cases} np & \textbf{for} \quad np \leq 1 \\ 1 & \textbf{for} \quad np > 1 \end{cases} \\
&\geq np - \frac{n(n-1)}{2}p^2 = np - \frac{(np)^2}{2} \\
\Pr(\bigcup_{j=1}^{n} B_j) &\geq \geq \begin{cases} np/2 & \textbf{for} \quad np \leq 1 \\ 1/2 & \textbf{for} \quad np > 1 \end{cases}
\end{aligned}$$

$$\Pr\left(\bigcup_{j=2}^{M}(W_j \geq w_1\right) \leq \begin{cases} (M-1)Q(w_1) & \textbf{for} \quad w_1 \geq \gamma \\ 1 & \textbf{for} \quad w_1 < \gamma \end{cases}$$

$$\begin{aligned} \Pr(e) &\leq \int_{-\infty}^{\gamma} f_{W_1}(w_1)\, dw_1 + \int_{\gamma}^{\infty} f_{W_1}(w_1)(M-1)Q(w_1)\, dw_1 \\ &= Q(\alpha-\gamma) + \int_{\gamma}^{\infty} \frac{M-1}{\sqrt{2\pi}} Q(w_1) \exp\left(\frac{-(w_1-\alpha)^2}{2}\right) \end{aligned}$$

Expression on right looks Gaussian, mean $\alpha/2$.

Bottom line: Choose $\gamma = \sqrt{2 \ln M}$ Then

$$\Pr(e) \leq \begin{cases} \exp\left(\frac{-(\alpha-\gamma)^2}{2}\right) & \textbf{for} \quad \alpha/2 \leq \gamma \\ \exp\left(\frac{-\alpha^2}{4} + \frac{\gamma^2}{2}\right) & \textbf{for} \quad \alpha/2 > \gamma \end{cases}$$

Let $\log M = b$ **and** $E_b = E/b$**. Then**

$$\Pr(e) \leq \begin{cases} \exp\left[-b\left(\sqrt{\mathbf{E}_b/N_0} - \sqrt{\ln 2}\right)^2\right] & \textbf{for} \quad \frac{E_b}{4N_0} \leq \ln 2 < \frac{E_b}{N_0} \\ \exp\left[-b\left(\frac{\mathbf{E}_b}{2N_0} - \ln 2\right)\right] & \textbf{for} \quad \ln 2 < \frac{E_b}{4N_0} \end{cases}$$

This says we can get arbitrarily small error probability so long as $E_b/N_0 > \ln 2$.

This is Shannon's capacity formula for unlimited bandwidth WGN transmission.

Bi-Orthogonal code by Hadamard matrix

Map n bit blocks to 2^n bit orthogonal sequences.

0	0
0	1

$b = 1$

0	0	0	0
0	1	0	1
0	0	1	1
0	1	1	0

$b = 2$

0000	0000
0101	0101
0011	0011
0110	0110
0000	1111
0101	1010
0011	1100
0110	1001

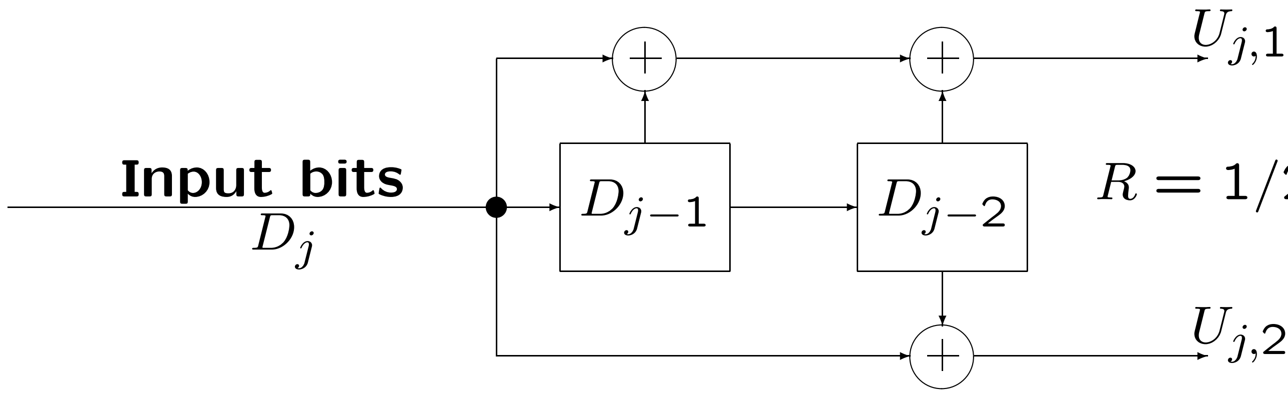
$b = 3$

Generate H_{b+1} from H_b: put H_b at top left, top right, lower left, and put complement $\overline{H}_b$ at lower right.

Each mod 2 row sum is a row - half ones.

Follow by antipodal modulation.

Convolutional Encoding

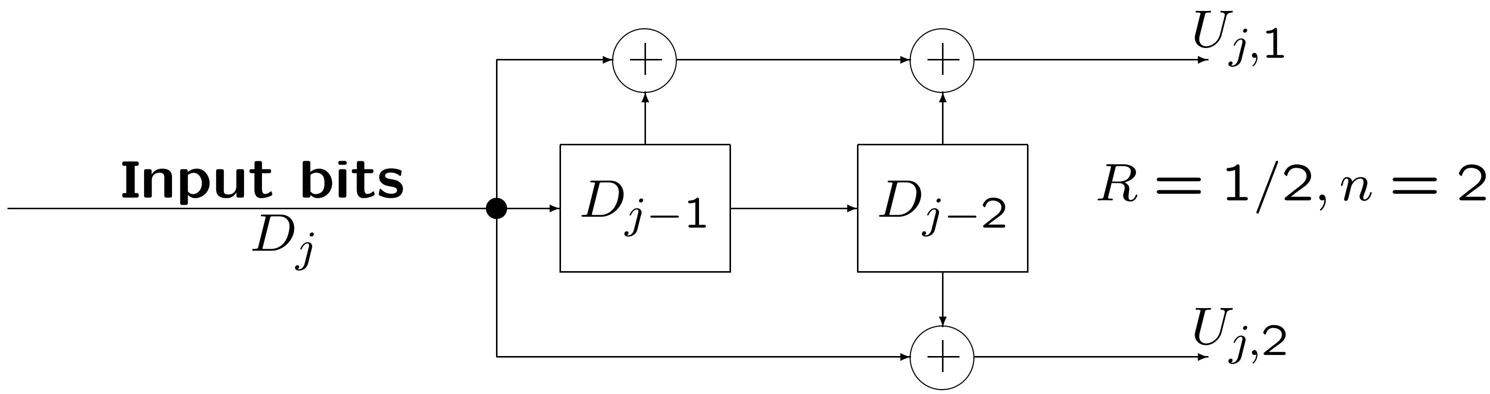


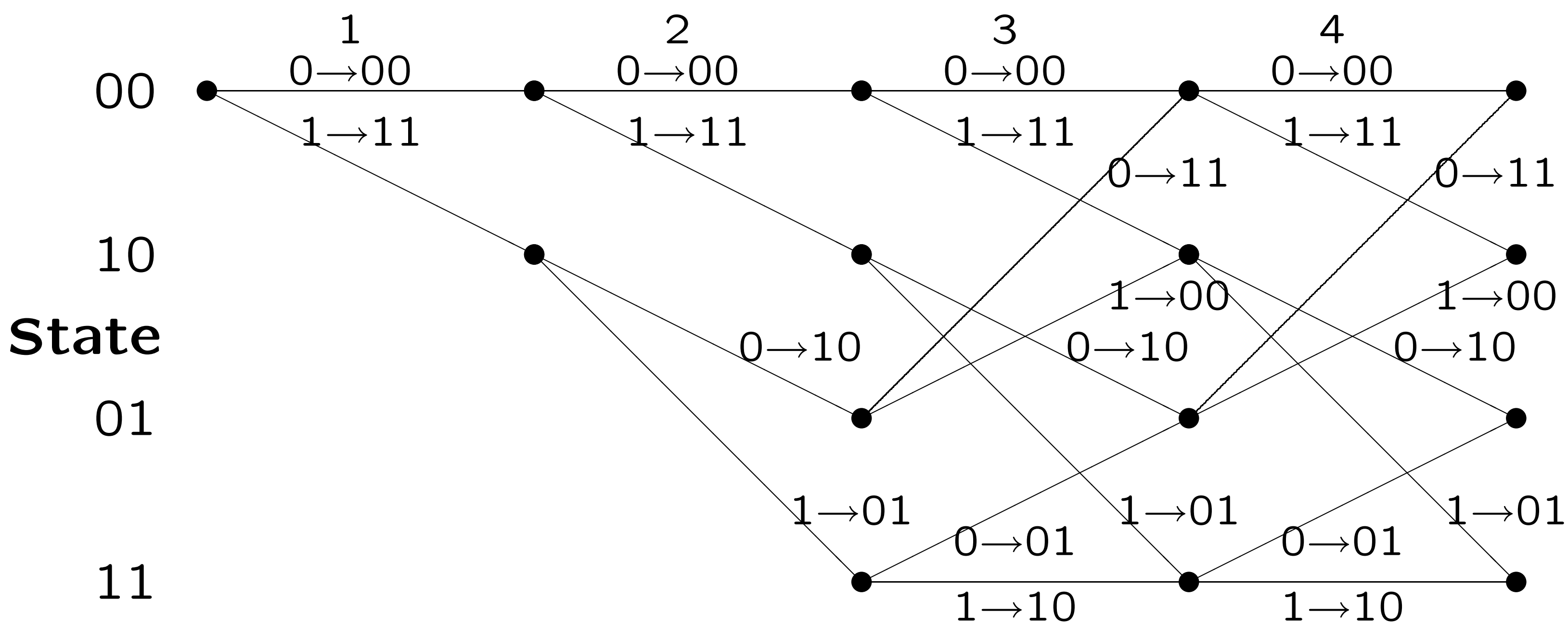
$R = 1/2, n = 2$

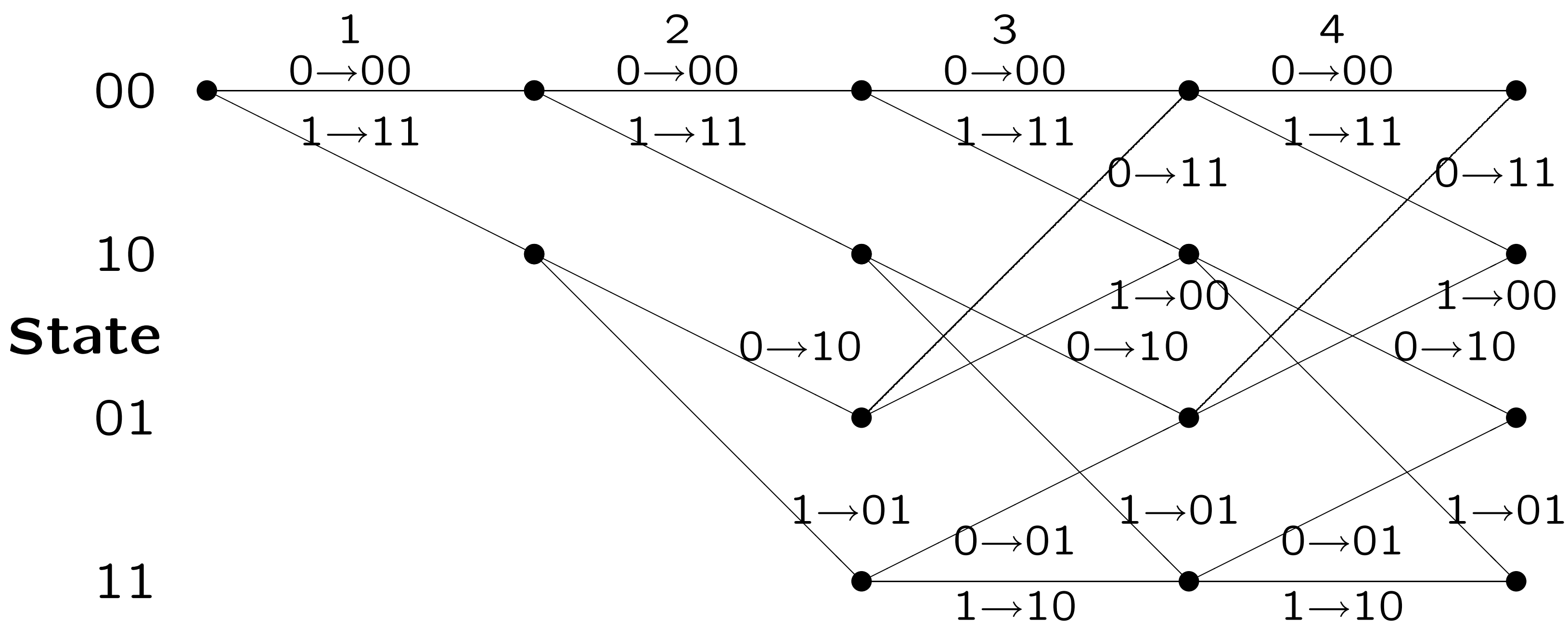
$$\begin{aligned} U_{j,1} &= D_j \oplus D_{j-1} \oplus D_{j-2} \\ U_{j,2} &= D_j \qquad\qquad \oplus D_{j-2} \end{aligned}$$

It needs n bits at end of block to return to state 0.

Viterbi algorithm used for decoding; complexity $\sim 2^n$.


Input bits
D_j
D_{j-1}
D_{j-2}
$U_{j,1}$
$U_{j,2}$
$R = 1/2, n = 2$


1
2
3
4
00
10
State
01
11
0→00
1→11
0→11
1→00
0→10
1→01
0→01
1→10



Viterbi decoding: At each epoch, decode conditional on each possible assumed state.

Maintain only the survivor at each state; each decoding step is a binary decision.

WIRELESS COMMUNICATION

- Wireless: radiation between antennas.
- Much more difficult than wires.
- Permits motion and temporary locations.
- Avoids mazes of wires

NEW PROBLEMS:

1. Channel changes with time
2. Interference between channels

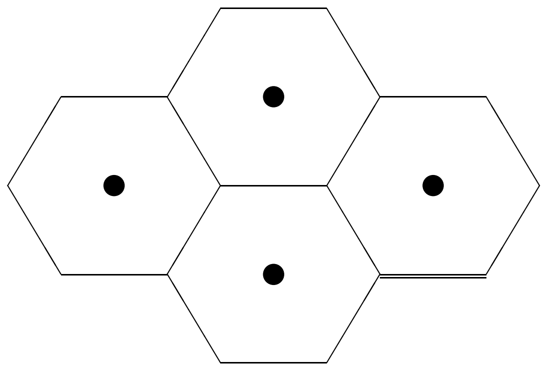
Started by Marconi in 1897; Many false starts

We will concentrate on Cellular Networks

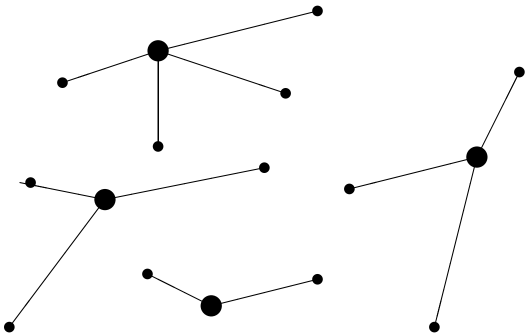
This includes most features of other systems.

Many mobiles, Few base stations.

Mobile $\rightarrow$ Base station $\rightarrow$ MTSO $\rightarrow$ Wired network $\rightarrow$ Whatever

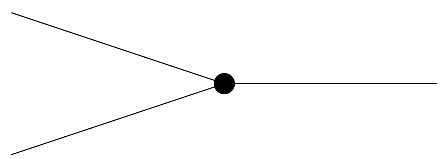


Hexagon Cells



Real Cells

Base Stations



MTSO

Cellular Network is Appendage to Wire Network

Major Problems:

- Outgoing: Find Best Base station
- Ingoing: Find Mobile
- Multiple mobiles send to same base station. This is called the reverse channel or a multiaccess channel
- Base station sends to multiple mobiles. this is called the forward channel or a broadcast channel.

Wireless Systems are now digital (Binary Interface)

Source either analog or digital.

Cellular systems developed for voice

But major issues quite different for voice and data

OTHER WIRELESS SYSTEMS:

Broadcast Systems

Wireless LANs (often in home or office)

Adhoc Networks

Standardization is a major problem for all wireless systems

Particularly a problem for cellular because of roaming.

Will voice and data wireless networks merge into one, or will they evolve into separate networks?

Is there a large market for high speed mobile data?

We study more technical issues in what follows.

PHYSICAL MODELING

Wireless uses bandwidths of KH to a few MH in bands of a few GH.

Cellular ranges are small, a few KM or less

Narrow band; WGN assumption good, but new problems are fading and interference.

EM equations are too difficult to solve and constantly changing.

Very different modeling questions arise in the placement of base stations from those in the design of mobiles and base stations.

Look at idealized models for clues

Consider fixed antenna in free space:

Response at $x = (r, \theta, \psi)$ to sinusoid at f:

$$E(f,t,x) = \frac{1}{r}\Re\left[\alpha_s(x,f))\,\mathsf{exp}\{2\pi i f(t - \frac{r}{c})\}\right]$$

Note $1/r$ attenuation; think spheres

Receiving antenna alters field; doesn't depend on (r, θ, ψ). Define

$$H(f) = \frac{\alpha(\theta,\psi,f)\,\mathsf{exp}\{-2\pi i f r/c\}}{r}$$

$$E_r(f,t,u) = \Re\,[H(f)\,\mathsf{exp}\{2\pi i f t\}]$$

Linearity holds but not time invariance.

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6.450 Principles of Digital Communication I
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