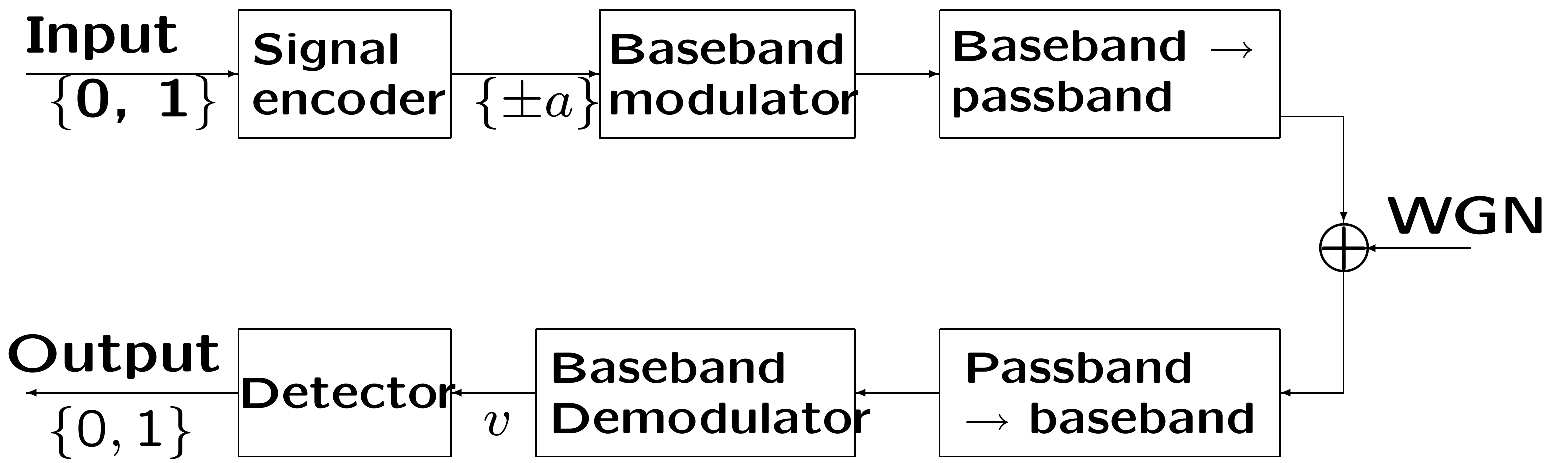
BINARY DETECTION



A detector observes a sample value of a rv V (or vector, or process) and guesses the value of another rv, H with values $\{0, 1\}$ (binary detection), $\{1, 2, \ldots, M\}$ (general detection).

Synonyms: hypothesis testing, decision making, decoding.

Assume that the detector is designed on the basis of a complete probability model.

That is, the joint probability distribution of H and V are known.

The objective is to maximize the probability of guessing correctly (i.e., to minimize the probability of error).

Let H be the rv to be detected (guessed) and V the rv to be observed.

The experiment is performed, $V = v$ is observed and $H = m$, is not observed; the detector chooses $\widetilde{H}(v) = j$, and an error occurs if $m \neq j$.

In principle, the problem is simple.

Given $V = v$, we calculate $p_{H|V}(m \,|\, v)$ for each m, $1 \le m \le M$.

This <u>is</u> the probability that m is correct conditional on v. The MAP (maximum a posteriori probability) rule is: choose $\widetilde{H}(v)$ to be that m for which $p_{H|V}(m \,|\, v)$ is maximized.

$$\widetilde{H}(v) = \arg\max_m [p_{H|V}(m \,|\, v)] \qquad \textbf{(MAP rule)},$$

The probability of being correct is $p_{H|V}(m \,|\, v)$ for that m. Averaging over v, we get the overall probability of being correct.

BINARY DETECTION

H takes the values 0 or 1 with probabilities p_0 and p_1. We assume initially that only one binary digit is being sent rather than a sequence.

Assume initially that the demodulator converts the received waveform into a sample value of a rv V with a probability density.

Usually the conditional densities $f_{V|H}(v \mid m)$, $m \in \{0, 1\}$ can be found from the channel characteristics.

These are called likelihoods. The marginal densiity of V is then

$$f_V(v) = p_0 f_{V|H}(v \mid 0) + p_1 f_{V|H}(v \mid 1)$$

$$p_{H|V}(j \,|\, v) = \frac{p_j f_{V|H}(v \,|\, j)}{f_V(v)}.$$

The MAP decision rule is

$$\frac{p_0 f_{V|H}(v \,|\, 0)}{f_V(v)} \ \underset{\tilde{H}=1}{\overset{\tilde{H}=0}{\substack{\geq \\ <}}} \ \frac{p_1 f_{V|H}(v \,|\, 1)}{f_V(v)}.$$

$$\Lambda(v) = \frac{f_{V|H}(v \,|\, 0)}{f_{V|H}(v \,|\, 1)} \ \underset{\tilde{H}=1}{\overset{\tilde{H}=0}{\substack{\geq \\ <}}} \ \frac{p_1}{p_0} = \eta.$$

For <u>any</u> binary detection problem where the observation is a sample value v of a random something with a probability density:

Calculate the likelihood ratio $\Lambda(v) = \frac{f(v|0)}{f(v|1)}$.

MAP: Compare $\Lambda(v)$ with threshold $\eta = p_1/p_0$.

If $\geq$, $\widetilde{H} = 0$; otherwise $\widetilde{H} = 1$.

MAP rule partitions V space into 2 regions.

Error occurs, for $H = m$, if v lands in the region for m complement.

MAP rule minimizes error probability.

Example: 2PAM in Gaussian noise.

$H=0$ **means** $+a$ **enters modulator;** $H=1$ **means** $-a$ **enters modulator.**

$V = \pm a + Z$, $Z \sim \mathcal{N}(0, N_0/2)$ **comes out of demodulator.**

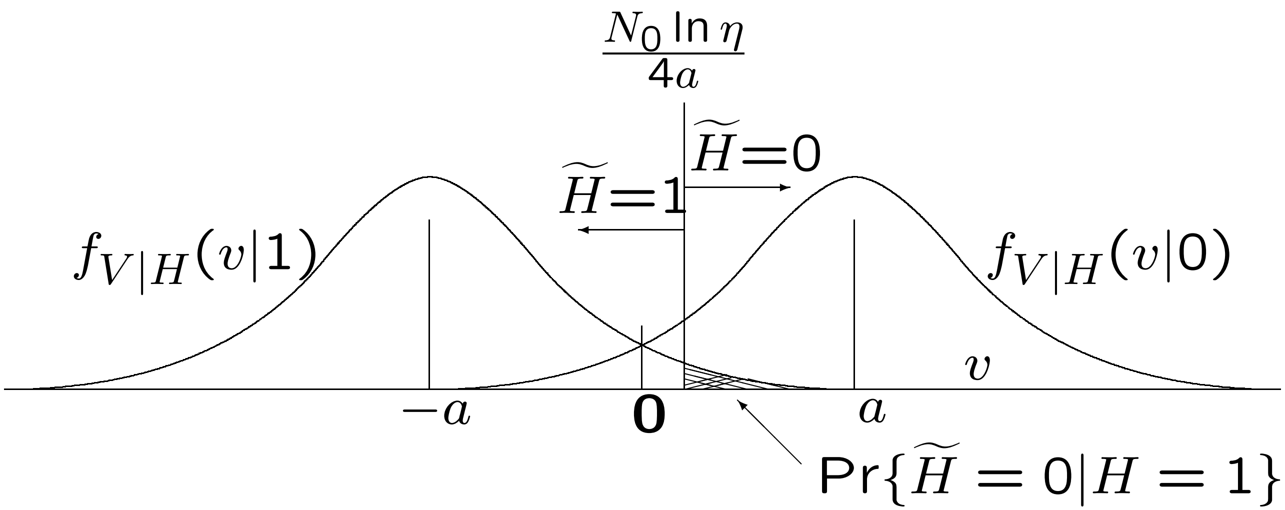
We only send one binary digit H**; the detector observes only** V**.**

$$\begin{aligned}
f_{V|H}(v\,|\,0) &= \frac{1}{\sqrt{\pi N_0}} \exp\left[\frac{-(v-a)^2}{N_0}\right] \\
f_{V|H}(v\,|\,1) &= \frac{1}{\sqrt{\pi N_0}} \exp\left[\frac{-(v+a)^2}{N_0}\right] \\
\Lambda(v) &= \exp\left[\frac{-(v-a)^2 + (v+a)^2}{N_0}\right] = \exp\left[\frac{4av}{N_0}\right].
\end{aligned}$$

$$\exp\left[\frac{4av}{N_0}\right] \quad \underset{\widetilde{H}=1}{\overset{\widetilde{H}=0}{\geq \atop <}} \quad \frac{p_1}{p_0} = \eta.$$

$$\mathbf{LLR}(v) = \left[\frac{4av}{N_0}\right] \quad \underset{\widetilde{H}=1}{\overset{\widetilde{H}=0}{\geq \atop <}} \quad \ln(\eta).$$

$$v \quad \underset{\widetilde{H}=1}{\overset{\widetilde{H}=0}{\geq \atop <}} \quad \frac{N_0 \ln(\eta)}{4a}.$$



For $H = 1$, error occurs if $Z \geq a + \frac{N_0 \ln \eta}{4a}$.

The larger $2a/N_0$, the less important η is.

$$\Pr\{e \mid H{=}1\} = Q\left(\frac{a}{\sqrt{N_0/2}} + \frac{\sqrt{N_0/2}\ln \eta}{2a}\right)$$

$$Q(x) = \int_x^\infty \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-z^2}{2}\right) dz.$$

For communication, usually assume $p_0 = p_1$ so $\eta = 1$.

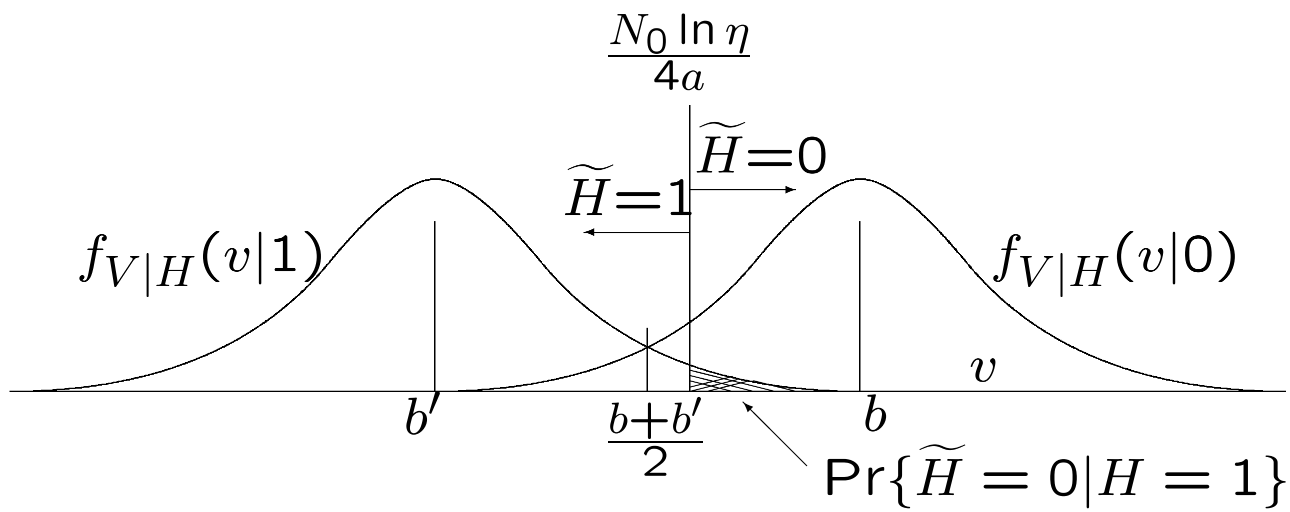
$$\Pr\{e\} = \Pr\{e \,|\, H{=}1\} = \Pr\{e \,|\, H{=}0\} = Q\left(\frac{a}{\sqrt{N_0/2}}\right)$$

The energy per bit is $E_b = a^2$, so

$$\Pr\{e\} = \Pr\{e \,|\, H{=}1\} = \Pr\{e \,|\, H{=}0\} = Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$$

This makes sense - only the ratio of E_b to N_0 can be relevant, since both can be scaled together.

Detection for binary non-antipodal signals:



This is the same as before if $2a = b - b'$**.**

View the center point $c = (b + b')/2$ **as a pilot tone or some other non-information baring signal with** $\pm a$ **added to it.**

$\Pr(e)$ **remains the same, but** $E_b = a^2 + c^2$**.**

REAL ANTIPODAL VECTORS IN WGN

$H{=}0 \rightarrow \ \vec{a}{=}(a_1, \dots, a_k)$ **and** $H{=}1 \rightarrow -\vec{a}{=}(-a_1, \dots, -a_k)$.

$$\vec{V} = \pm\vec{a} + \vec{Z}$$

where $\vec{Z} = (Z_1, \dots, Z_k)$**, iid,** $Z_j \sim \mathcal{N}(0, N_0/2)$.

$$f_{\vec{V}|H}(\vec{v} \mid 0) = \frac{1}{(\pi N_0)^{k/2}} \exp\left(\frac{-\|\vec{v}-\vec{a}\|^2}{N_0}\right).$$

$$\text{LLR}(\vec{v}) = \frac{-\|\vec{v}-\vec{a}\|^2 + \|\vec{v}+\vec{a}\|^2}{N_0} = \frac{4\langle\vec{v}, \vec{a}\rangle}{N_0}$$

The MAP decision compares this with $\ln\eta = \ln\left(\frac{p_1}{p_0}\right)$.

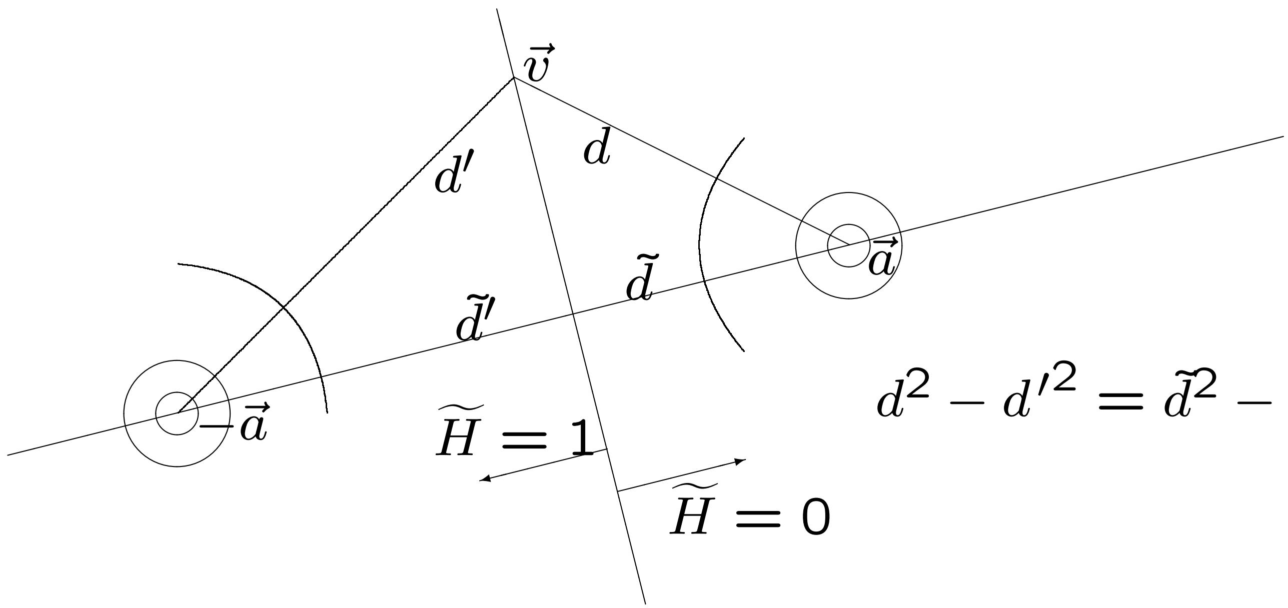
We call $\langle\vec{v}, \vec{a}\rangle$ **a sufficient statistic (something from which** $\Lambda(\vec{v})$ **can be calculated.**

$$\text{LLR}(\vec{v}) = \frac{-\|\vec{v} - \vec{a}\|^2 + \|\vec{v} + \vec{a}\|^2}{N_0} = \frac{4\langle \vec{v}, \vec{a} \rangle}{N_0}$$

Since the scalar $\langle \vec{v}, \vec{a} \rangle$ is a sufficient statistic, the problem is reduced to the scalar binary detection problem.

The vector problem reduces to scalar 2PAM by interpreting $\langle \vec{v}, \vec{a} \rangle$ as the observation, which is $Z \pm \|\vec{a}\|$.

Each component of the received vector is weighted by the corresponding signal component.



$$d^2 - d'^2 = \tilde{d}^2 - \tilde{d}'^2$$

The probability of error, with $\eta = 1$, is

$$\Pr\{e\} = Q\left(\frac{\|\vec{a}\|}{\sqrt{N_0/2}}\right) = Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$$

Summary of binary detection with vector observation in iid Gaussian noise.:

First remove center point from signal and its effect on observation.

Then signal is $\pm\vec{a}$. and $\vec{v} = \pm\vec{a} + \vec{Z}$.

Find $\langle \vec{v}, \vec{a} \rangle$ and compare with threshold (0 for ML case).

This does not depend on the vector basis - becomes trivial if $\vec{a}$ normalized is a basis vector.

Received components orthogonal to signal are irrelevant.

This same argument is valid for waveforms if we expand them in an orthonormal expansion. Then the modulated signal is a vector and the noise is a vector.

There is a funny mathematical issue:

$$f_{\vec{Y}|H}(\vec{y}|0) = \lim_{k\to\infty} \frac{1}{[2\pi(N_0/2)]^{k/2}} \exp \frac{-\|\vec{y}-\vec{a}\|^2}{N_0}$$

This doesn't converge, but $\langle \vec{v}, \vec{a} \rangle$ converges.

In other words, the fact that the noise is irrelevant outside of the range of the signal makes it unnecessary to be careful about modeling the noise there.

Consider binary PAM with pulse shape $p(t)$ Suppose only one signal sent, so $u(t) = \pm a\, p(t)$.

As a vector, $\vec{u} = \pm a\vec{p}$. The receiver calculates $\langle \vec{v}, a\vec{p} \rangle$.

This says that the MAP detector is a matched filter followed by sampling and a one dimensional threshold detector.

Using a square root of Nyquist pulse with a matched filter avoids intersymbol interference and minimizes error probability (for now in absence of other signals).

Complex antipodal vectors in WGN.

$\vec{u} = (u_1, \dots, u_k)$ where for each j, $u_j \in \mathbb{C}$. $H{=}0 \to \vec{u}$ and $H{=}1 \to -\vec{u}$.

Let $\vec{Z} = (Z_1, \dots, Z_k)$ be a vector of k zero-mean complex iid Gaussian rv's with iid real and imaginary parts, each $\mathcal{N}(0, N_0/2)$.

Under $H{=}0$, the observation $\vec{V}$ is given by $\vec{V} = \vec{u} + \vec{Z}$; under $H{=}1$, $\vec{V} = -\vec{u} + \vec{Z}$.

let $\vec{a}$ be the $2k$ dimensional real vector with components $\Re(u_j)$ and $\Im(u_j)$ for $1 \le j \le k$.

Let $\vec{Z}'$ be the $2k$ dimensional real random vector with components $\Re(Z_j)$ and $\Im(Z_j)$ for $1 \le j \le k$.

$$f_{\vec{Y}|H}(\vec{y} \mid 0) = \frac{1}{(\pi N_0)^k} \exp \sum_{j=1}^{2k} \frac{-(y_j - a_j)^2}{N_0}$$

$$= \frac{1}{(\pi N_0)^k} \exp \frac{-\|\vec{y} - \vec{a}\|^2}{N_0}.$$

$$\frac{\langle \vec{y}, \vec{a} \rangle}{\|\vec{a}\|} \quad \begin{matrix} \tilde{H}=0 \\ \geq \\ < \\ \tilde{H}=1 \end{matrix} \quad \frac{N_0 \ln(\eta)}{4\|\vec{a}\|}$$

.

$$
\begin{aligned}
\langle \vec{y}, \vec{a} \rangle &= \sum_{j=1}^{k} \left[\Re(v_j)\Re(u_j) + \Im(v_j)\Im(u_j) \right] \\
&= \sum_{j=1}^{k} \Re(v_j u_j^*) = \Re(\langle \vec{v}, \vec{u} \rangle).
\end{aligned}
$$

Thus one has to take the real part of $\langle \vec{v}, \vec{u} \rangle$**.**