

Review of zero-mean jointly Gaussian $\mathbf{r}\vec{v}$'s:

A random vector $\vec{Z} = (Z_1, \dots, z_k)^\top$ of linearly independent rv's is jointly Gauss iff

1. $\vec{Z} = \mathbf{A}\vec{N}$ for normal $\mathbf{r}\vec{v}$ \vec{N} ,
2. $f_{\vec{Z}}(\vec{z}) = \frac{1}{(2\pi)^{k/2} \sqrt{|\det(\mathbf{K}_{\vec{Z}})|}} \exp \left[-\frac{1}{2} \vec{z}^\top \mathbf{K}_{\vec{Z}}^{-1} \vec{z} \right]$.
3. $f_{\vec{Z}}(\vec{z}) = \prod_{j=1}^k \frac{1}{\sqrt{2\pi\lambda_j}} \exp \left[\frac{-|\langle \vec{z}, \vec{q}_j \rangle|^2}{2\lambda_j} \right]$ for $\{\vec{q}_j\}$ orthonormal, $\{\lambda_j\}$ positive.
4. All linear combinations of \vec{Z} are Gaussian.

$Z(t)$ is a Gaussian process if $Z(t_1), \dots, Z(t_k)$ is jointly Gauss for all k and $\{t_i\}$.

A general way to generate a Gaussian process is to start with a sequence of bounded orthonormal functions, $\phi_1(t), \phi_2(t), \dots$ and a sequence Z_1, Z_2, \dots of independent Grv's such that $\sum_j \mathbf{E}[|Z_j|^2] < \infty$.

Then $Z(t) = \sum_j Z_j \phi_j(t)$ is a Gaussian random process. Also the sample functions of $Z(t)$ are \mathcal{L}_2 with probability 1.

Assume sample functions are \mathcal{L}_2 from now on.

A linear functional is a rv given by

$$V = \int Z(t)g(t) dt.$$

This means that for all $\omega \in \Omega$,

$$V(\omega) = \langle Z(t, \omega), g(t) \rangle = \int_{-\infty}^{\infty} Z(t, \omega)g(t) dt.$$

If $Z(t) = \sum_j Z_j \phi_j(t)$, then

$$V = \sum_j Z_j \langle \phi_j, g \rangle$$

We won't worry about the details of convergence here, but V turns out to be Gaussian with finite variance.

Let $Z(t) = \sum_j Z_j \phi_j(t)$ be a zero-mean Gaussian process where Z_1, Z_2, \dots are independent Gauss and ϕ_1, ϕ_2, \dots , are orthonormal.

Let $g_1(t), \dots, g_k(t)$ be \mathcal{L}_2 functions. Then the linear functionals

$$V_\ell = \int Z(t) g_\ell(t) dt$$

are zero-mean jointly Gaussian.

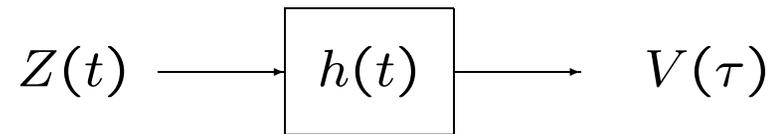
Proof: All linear combinations are Gaussian.

The covariance matrix of \vec{V} has elements

$$\mathbf{E}[V_i V_\ell] = \int \int g_i(t) \mathbf{K}_{\vec{Z}}(t, \tau) g_\ell(\tau) dt d\tau$$

where $\mathbf{K}_{\vec{Z}}(t, \tau) = \sum_j \mathbf{E}[|Z_j|^2] \phi_j(t) \phi_j(\tau)$.

LINEAR FILTERING OF PROCESSES



$$\begin{aligned} V(\tau, \omega) &= \int_{-\infty}^{\infty} Z(t, \omega) h(\tau - t) dt \\ &= \sum_j Z_j(\omega) \int_{-\infty}^{\infty} \phi_j(t) h(\tau - t) dt. \end{aligned}$$

For each τ , this is sample value of a linear functional. For any τ_1, \dots, τ_k , $V(\tau_1), \dots, V(\tau_k)$ are jointly Gaussian.

If $\{Z(t)\}$ is a Gaussian process, $\{V(t)\}$ is also.

This is an alternate way to generate Gaussian processes.

The covariance function of a filtered process follows in the same way as the covariance matrix for linear functionals.

$$\begin{aligned}\mathbf{K}_{\vec{V}}(r, s) &= \mathbf{E}[V(r)V(s)] \\ &= E \left[\int Z(t)h(r-t) dt \int Z(\tau)h(s-\tau) d\tau \right] \\ &= \int \int h(r-t)\mathbf{K}_{\vec{Z}}(t, \tau)h(s-\tau) dt d\tau\end{aligned}$$

This looks like the matrix equations we had before.

STATIONARY RANDOM PROCESSES:

$\{Z(t); t \in \mathbb{R}\}$ is stationary if $Z(t_1), \dots, Z(t_k)$ and $Z(t_1 + \tau), \dots, Z(t_k + \tau)$ have same distribution for all τ , all k , and all t_1, \dots, t_k .

Stationary implies that $E[Z(t)] = c$ for all t and

$$E[Z(t_1)Z(t_2)] = E[Z(t_1 - t_2)Z(0)]$$

for all t_1, t_2 . That is, $E[Z(t)] = c$ for all t and

$$\mathbf{K}_{\vec{Z}}(t_1, t_2) = \mathbf{K}_{\vec{Z}}(t_1 - t_2, 0) = \tilde{\mathbf{K}}_{\vec{Z}}(t_1 - t_2).$$

Note that $\tilde{\mathbf{K}}_{\vec{Z}}(t)$ is real and symmetric.

Assume zero-mean from now on.

A process is wide sense stationary (WSS) if $\mathbf{E}[Z(t)] = \mathbf{E}[Z(0)]$ and $\mathbf{K}_{\vec{Z}}(t_1, t_2) = \mathbf{K}_{\vec{Z}}(t_1 - t_2, 0)$ for all t, t_1, t_2 . Define

$$\tilde{\mathbf{K}}_{\vec{Z}}(t) = \mathbf{K}_{\vec{Z}}(t, 0)$$

for WSS processes.

This is symmetric and is maximized at $t = 0$.

A Gaussian process is stationary if it is WSS.

An important example is $V(t) = \sum_k V_k \text{sinc}\left(\frac{t-kT}{T}\right)$.

If $\mathbf{E}[V_k V_i] = \sigma^2 \delta_{i,k}$, then

$$\mathbf{K}_{\vec{V}}(t, \tau) = \sigma^2 \sum_k \text{sinc}\left(\frac{t-kT}{T}\right) \text{sinc}\left(\frac{\tau-kT}{T}\right).$$

Then $\{V(t); t \in \mathbb{R}\}$ is WSS with

$$\tilde{\mathbf{K}}_{\vec{V}}(t - \tau) = \sigma^2 \text{sinc}\left(\frac{t - \tau}{T}\right).$$

Proof: Expand baseband limited $u(t)$ as

$$u(t) = \sum_k u(kT) \text{sinc}\left(\frac{t-kT}{T}\right)$$

$$u(t) = \sum_k u(kT) \operatorname{sinc} \left(\frac{t - kT}{T} \right)$$

For given τ , let $u(t) = \operatorname{sinc}(\frac{t-\tau}{T})$. Substituting,

$$\begin{aligned} \operatorname{sinc} \left(\frac{t-\tau}{T} \right) &= \sum_k \operatorname{sinc} \left(\frac{kT-\tau}{T} \right) \operatorname{sinc} \left(\frac{t-kT}{T} \right) \\ &= \sum_k \operatorname{sinc} \left(\frac{\tau-kT}{T} \right) \operatorname{sinc} \left(\frac{t-kT}{T} \right) \end{aligned}$$

$$\begin{aligned} \mathbf{K}_{\vec{V}}(t, \tau) &= \sigma^2 \sum_k \operatorname{sinc} \left(\frac{t - kT}{T} \right) \operatorname{sinc} \left(\frac{\tau - kT}{T} \right) \\ &= \sigma^2 \operatorname{sinc} \left(\frac{t-\tau}{T} \right) \end{aligned}$$

$$V(t) = \sum_k V_k \operatorname{sinc}\left(\frac{t - kT}{T}\right)$$

Example 1: Let the V_k be iid binary antipodal. Then $v(t)$ is WSS, but not stationary.

Example 2: Let the V_k be iid zero-mean Gauss. then $V(t)$ is WSS and stationary (and zero-mean Gaussian).

The sample functions of a WSS non-zero process are not \mathcal{L}_2 .

The covariance $\tilde{\mathbf{K}}_{\vec{V}}(t)$ is \mathcal{L}_2 in cases of physical relevance. It has a Fourier transform called the spectral density.

$$S_{\vec{V}}(f) = \int \tilde{\mathbf{K}}_{\vec{V}}(t) e^{-2\pi i f t} dt$$

The spectral density is real and symmetric.

Consider a set of linear functions:

$$V_j = \int Z(t)g_j(t) dt.$$

Then

$$\begin{aligned}\mathbf{E}[V_i V_j] &= \mathbf{E} \left[\int_{-\infty}^{\infty} Z(t)g_i(t) dt \int_{-\infty}^{\infty} Z(\tau)g_j(\tau) d\tau \right] \\ &= \int_{t=-\infty}^{\infty} \int_{\tau=-\infty}^{\infty} g_i(t) \mathbf{E}[Z(t)Z(\tau)] g_j(\tau) dt d\tau \\ &= \int_{t=-\infty}^{\infty} \int_{\tau=-\infty}^{\infty} g_i(t) \mathbf{K}_{\vec{Z}}(t, \tau) g_j(\tau) dt d\tau\end{aligned}$$

If $\{Z(t); t \in \mathbb{R}\}$ is WSS,

$$\mathbf{E}[V_i V_j] = \int_{t=-\infty}^{\infty} g_i(t) \tilde{\mathbf{K}}_{\vec{Z}}(t - \tau) g_j(\tau) dt d\tau$$

$$\begin{aligned}
\mathbf{E}[V_i V_j] &= \int_{t=-\infty}^{\infty} g_i(t) \tilde{\mathbf{K}}_{\vec{Z}}(t - \tau) g_j(\tau) dt d\tau \\
&= \int_{t=-\infty}^{\infty} g_i(t) [\tilde{\mathbf{K}}_{\vec{Z}} * \vec{g}_j](t) dt
\end{aligned}$$

Let $\theta(t) = [\tilde{\mathbf{K}}_{\vec{Z}}(t) * \vec{g}_j](t)$ be the convolution of $\tilde{\mathbf{K}}_{\vec{Z}}$ and \vec{g}_j . Since $\theta(t)$ is real, $\theta(t) = \theta^*(t)$. Using Parseval's theorem for Fourier transforms,

$$\mathbf{E}[V_i V_j] = \int_{-\infty}^{\infty} g_i(t) \theta^*(t) dt = \int_{-\infty}^{\infty} \hat{g}_i(f) \hat{\theta}^*(f) df$$

Since $\hat{\theta}(f) = S_{\vec{Z}}(f) \hat{g}_2(f)$,

$$\mathbf{E}[V_i V_j] = \int \hat{g}_i(f) S_{\vec{Z}}(f) \hat{g}_j^*(f) df$$

$$\mathbf{E}[V_i V_j] = \int \hat{g}_i(f) S_{\vec{Z}}(f) \hat{g}_j^*(f) df$$

If $\hat{g}_i(f)$ and $\hat{g}_j(f)$ do not overlap in frequency, then $\mathbf{E}[V_i V_j] = 0$.

This means that for a WSS process, no linear functional in one frequency band is correlated with any linear functional in another band.

For a Gaussian stationary process, all linear functionals in one band are independent of all linear functionals in any other band.

For Gaussian stationary processes, different frequency bands contain independent noise.

Suppose $V = \int g(t)Z(t) dt$. Then

$$\mathbf{E}[|V|^2] = \int \hat{g}(f)S_{\vec{Z}}(f)\hat{g}^*(f) df$$

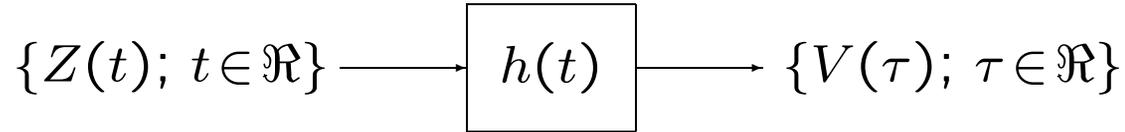
If $\hat{g}(f)$ has unit energy and is so narrow band that $S_{\vec{Z}}(f)$ is constant over region where $\hat{g}(f)$ is non-zero, then

$$\mathbf{E}[|V|^2] = S_{\vec{Z}}(f)$$

This means that $S(f)$ is the noise power per degree of freedom at frequency f .

Stationary Gaussian noise is independent between frequencies, and for narrow bandwidths, independent between orthonormal functionals.

LINEAR FILTERING OF PROCESSES



$$V(\tau) = \int_{-\infty}^{\infty} Z(t)h(\tau - t) dt.$$

For each τ , $V(\tau)$ is a linear functional. If $Z(t)$ is stationary, then

$$\begin{aligned} \tilde{\mathbf{K}}_{\vec{V}}(t-\tau) &= \int \left[\int h(t-\tau-\mu-\phi) \tilde{\mathbf{K}}_{\vec{Z}}(\phi) d\phi \right] h^*(-\mu) d\mu \\ &= (\vec{h}(t) * \tilde{\mathbf{K}}) * \vec{h}^*(-t) \\ S_{\vec{V}}(f) &= S_{\vec{Z}}(f) |\hat{h}(f)|^2 \end{aligned}$$

White noise is noise that is stationary over a large enough frequency band to include all frequency intervals of interest, i.e., $S_{\vec{z}}(f)$ is constant in f over all frequencies of interest.

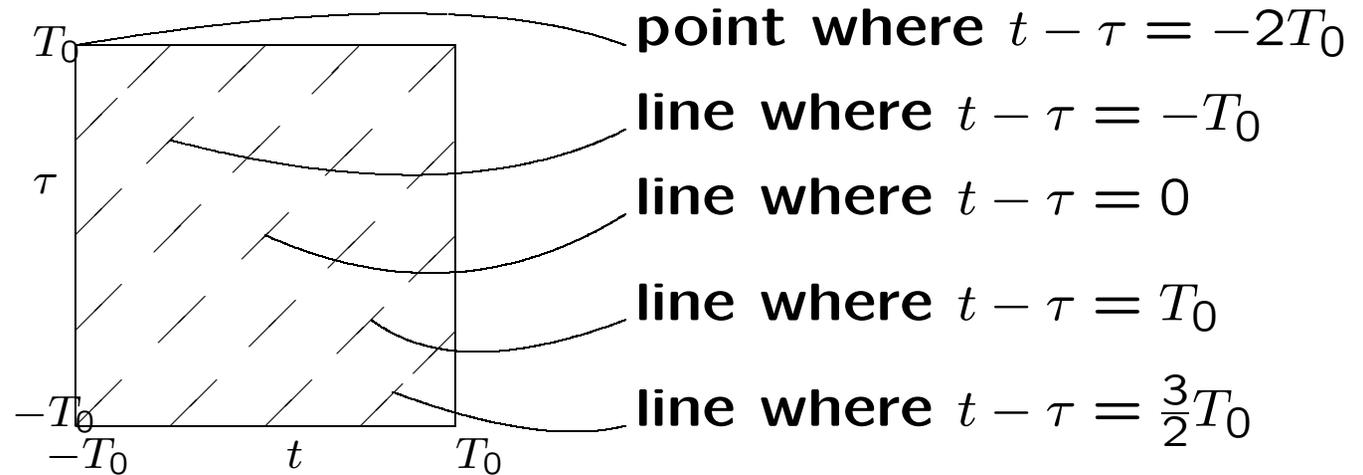
Within that frequency interval, $S_{\vec{z}}(f)$ can be taken for many purposes as $N_0/2$ and $\tilde{\mathbf{K}}_{\vec{z}}(t) = \frac{N_0}{2}\delta(t)$.

It is important to always be aware that this doesn't apply for frequencies outside the band of interest and doesn't make physical sense over all frequencies.

If the process is also Gaussian, it is called white Gaussian noise (WGN).

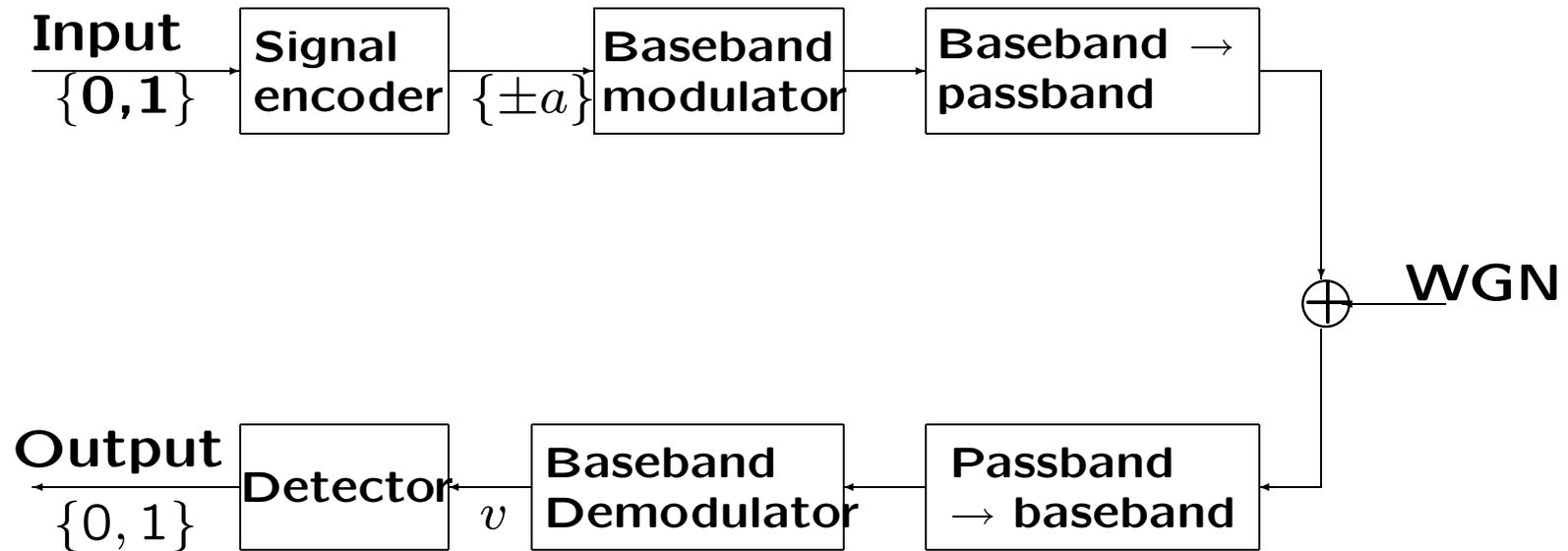
Definition: A zero-mean random process is effectively stationary (effectively WSS) within $[-T_0, T_0]$ if the joint probability assignment (covariance matrix) for t_1, \dots, t_k is the same as that for $t_1 + \tau, t_2 + \tau, \dots, t_k + \tau$ whenever t_1, \dots, t_k and $t_1 + \tau, t_2 + \tau, \dots, t_k + \tau$ are all contained in the interval $[-T_0, T_0]$.

A process is effectively WSS within $[-T_0, T_0]$ if $\mathbf{K}_{\vec{Z}}(t, \tau) = \tilde{\mathbf{K}}_{\vec{Z}}(t - \tau)$ for $t, \tau \in [-T_0, T_0]$.



Note that $\tilde{\mathbf{K}}_{\vec{z}}(t - \tau)$ for $t, \tau \in [-T_0, T_0]$ is defined in the interval $[-2T_0, 2T_0]$.

DETECTION



A detector observes a sample value of a rv V (or vector, or process) and guesses the value of another rv, H with values $0, 1, \dots, m-1$.

Synonyms: hypothesis testing, decision making, decoding.

We assume that the detector uses a known probability model.

We assume the detector is designed to maximize the probability of guessing correctly (i.e., to minimize the probability of error).

Let H be the rv to be detected (guessed) and V the rv to be observed.

The experiment is performed, $V = v$ is observed and $H = j$, is not observed; the detector chooses $\hat{H}(v) = i$, and an error occurs if $i \neq j$.

In principle, the problem is simple.

Given $V = v$, we calculate $p_{H|V}(j | v)$ for each j , $0 \leq j \leq m - 1$.

This is the probability that j is the correct conditional on v . The MAP (maximum a posteriori probability) rule is: choose $\hat{H}(v)$ to be that j for which $p_{H|V}(j | v)$ is maximized.

$$\hat{H}(v) = \arg \max_j [p_{H|V}(j | v)] \quad \text{(MAP rule),}$$

The probability of being correct is $p_{H|V}(j | v)$ for that j . Averaging over v , we get the overall probability of being correct.

BINARY DETECTION

H takes the values 0 or 1 with probabilities p_0 and p_1 . We assume initially that only one binary digit is being sent rather than a sequence.

Assume initially that the demodulator converts the received waveform into a sample value of a rv with a probability density.

Usually the conditional densities $f_{V|H}(v | j)$, $j \in \{0, 1\}$ can be found.

These are called likelihoods. The marginal density of V is then

$$f_V(v) = p_0 f_{V|H}(v | 0) + p_1 f_{V|H}(v | 1)$$

$$p_{H|V}(j | v) = \frac{p_j f_{V|H}(v | j)}{f_V(v)}.$$

The MAP decision rule is

$$\frac{p_0 f_{V|H}(v | 0)}{f_V(v)} \underset{\hat{H}=1}{\overset{\hat{H}=0}{\geq}} \frac{p_1 f_{V|H}(v | 1)}{f_V(v)}.$$

$$\Lambda(v) = \frac{f_{V|H}(v | 0)}{f_{V|H}(v | 1)} \underset{\hat{H}=1}{\overset{\hat{H}=0}{\geq}} \frac{p_1}{p_0} = \eta.$$

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