A random process $\{Z(t)\}$ is a collection of rv's, one for each $t \in \mathbb{R}$.

For any given epoch $t \in \mathbb{R}$, $Z(t)$ is a rv. It maps each $\omega \in \Omega$ into a real number $Z(t, \omega)$.

For any given $\omega \in \Omega$, $\{z(t); t \in \mathbb{R}\}$ is a sample function. It maps each $t \in \mathbb{R}$ into a real number $Z(t, \omega)$.

A random process is defined by a rule establishing a joint density $f_{Z(t_1),\dots,Z(t_k)}(z_1, \dots, z_k)$ for all k, $t_1, \dots, t_k$ and $z_1, \dots, z_k$.

Our favorite way to do this is $Z(t) = \sum Z_i \phi_i(t)$.

Joint densities on $Z_1, Z_2, \dots$ define $Z(t)$.

GAUSSIAN VARIABLES

Normalized Gaussian rv has density

$$f_N(n) = \frac{1}{\sqrt{2\pi}} \exp\left[\frac{-n^2}{2}\right].$$

Arbitrary Grv Z is shift by $\overline{Z}$, scale by σ^2

$$f_Z(z) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[\frac{-(z-\bar{Z})^2}{(2\sigma^2)}\right]$$

We describe the distribution of this Grv as $\mathcal{N}(\overline{Z}, \sigma^2)$

Refer to a k-tuple of rv's as $\vec{Z} = \{Z_1, \dots, Z_k\}$.

The set of k-tuples of rv's over a sample space is a vector space (but not the vector space $\mathbb{R}^{(k)}$ of real k-tuples).

Here we only want to use vector notation rather than any vector properties.

If $N_1, \dots, N_k$ are iid $\mathcal{N}(0,1)$, then joint density is

$$
\begin{aligned}
f_{\vec{N}}(\vec{n}) &= \frac{1}{(2\pi)^{k/2}} \exp\left(\frac{-n_1^2 - n_2^2 - \cdots - n_k^2}{2}\right) \\
&= \frac{1}{(2\pi)^{k/2}} \exp\left(\frac{-\|\vec{n}\|^2}{2}\right).
\end{aligned}
$$

Note spherical symmetry.

A k-tuple $\vec{Z}$ of rv's is zero-mean jointly Gaussian if, for real a_{ij}, and for iid $\mathcal{N}(0,1)$ rv's $\{N_1, \ldots, N_m\}$,

$$Z_i = \sum_{j=1}^{m} a_{ij} N_j$$

i.e., $\vec{Z}$ is zero-mean jointly Gauss if $\vec{Z} = \mathbf{A}\vec{N}$.

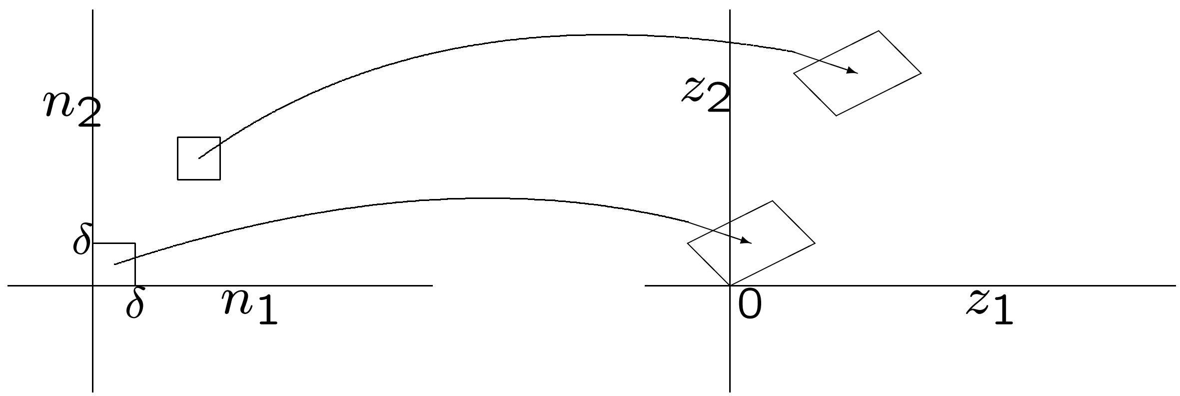
Jointly Gauss is more restrictive than individually Gauss; must be linear combinations of iid $\mathcal{N}(0,1)$.

Jointly Gauss is more general than independent Gauss.

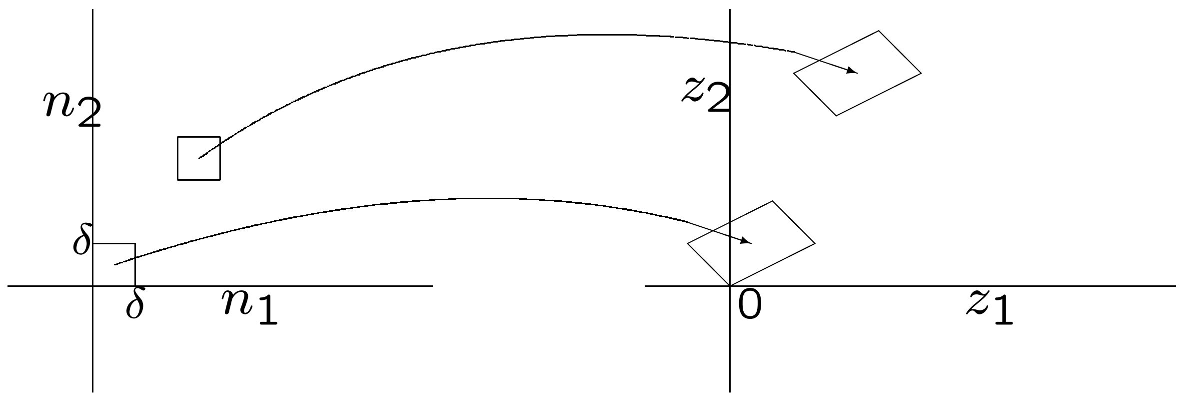
Think of $\vec{z} = \mathbf{A}\vec{n}$ in terms of sample values and take $m = k$.

$\mathbf{A}\vec{e}_j$ maps $\vec{e}_j$ into jth column of A.

Thus unit cube is mapped into parallelepiped whose edges are the columns of A.



$$Z_1 = N_1 + N_2 \text{ and } Z_2 = N_1 + 2N_2$$



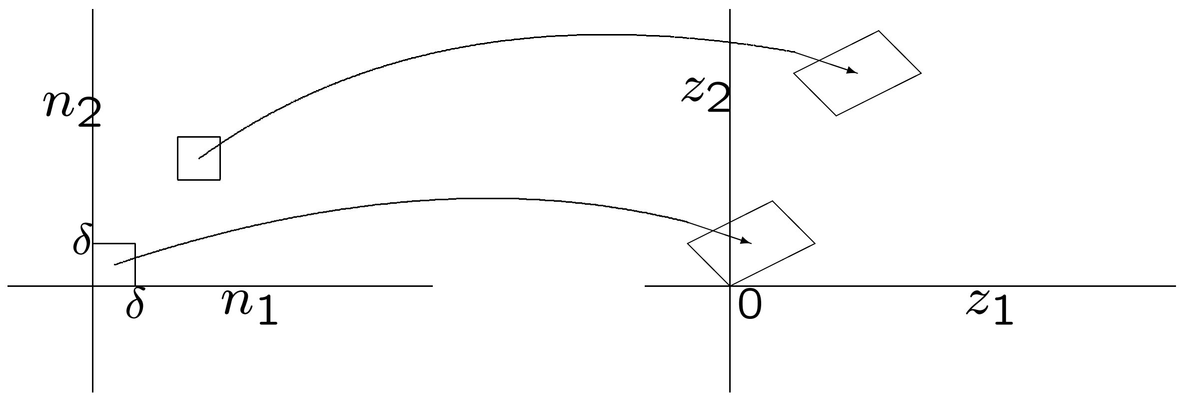
The mapping $\vec{z} = \mathbf{A}\vec{n}$ maps the unit cube $[0, \delta], \ldots [0, \delta]$ into the parallelepiped with sides $[0, \delta\vec{a}_1], \ldots, [0, \delta\vec{a}_m]$.

The volume of this parallelepiped is $|\det \mathbf{A}|$.

It maps $[n_1, n_1{+}\delta], \ldots [n_m, n_m{+}\delta]$ into

$$[n_1\vec{a}_1, (n_1{+}\delta)\vec{a}_1], \ldots, [n_m\vec{a}_m, (n_m{+}\delta)\vec{a}_m]$$

Assuming that A is non-singular, the mapping is invertible.



The probability of any given cube equals the probability of the corresponding parallelepiped.

$$f_{\vec{N}}(\vec{n})\delta^n \approx f_{\vec{Z}}(\vec{z})\delta^n \det \mathbf{A}$$

where $\det \mathbf{A}$ **is the volume of the parallelepiped with sides** $\vec{a}_1, \ldots, \vec{a}_m$**. Going to the limit** $\delta \to 0$**,**

$$f_{\vec{Z}}(\mathbf{A}\vec{n}) = \frac{f_{\vec{N}}(\vec{n})}{|\det \mathbf{A}|}.$$

$$f_{\vec{Z}}(\mathbf{A}\vec{n}) = \frac{f_{\vec{N}}(\vec{n})}{|\det \mathbf{A}|} \qquad f_{\vec{Z}}(\vec{z}) = \frac{f_{\vec{N}}(\mathbf{A}^{-1}\vec{z})}{|\det \mathbf{A}|}$$

$$\begin{aligned} \vec{f}_{\vec{Z}}(\vec{z}) &= \frac{1}{(2\pi)^{k/2}|\det(\mathbf{A})|} \exp\left(\frac{-\|\mathbf{A}^{-1}\vec{z}\|^2}{2}\right) \\ &= \frac{1}{(2\pi)^{k/2}|\det(\mathbf{A})|} \exp\left[-\frac{1}{2}\vec{z}^{\mathsf{T}}(\mathbf{A}^{-1})^{\mathsf{T}}\mathbf{A}^{-1}\vec{z}\right] \end{aligned}$$

For zero-mean rv's, covariance of Z_1, Z_2 is E$[Z_1 Z_2]$.

For m-tuple $\vec{Z}$, covariance is matrix whose i, j element is E$[Z_i Z_j]$. That is

$$\mathsf{K}_{\vec{Z}} = \mathsf{E}[\vec{Z}\vec{Z}^{\mathsf{T}}].$$

For $\vec{Z} = \mathsf{A}\vec{N}$, this becomes

$$\mathsf{K}_{\vec{Z}} = \mathsf{E}[\mathsf{A}\vec{N}\vec{N}^{\mathsf{T}}\mathsf{A}^{\mathsf{T}}] = \mathsf{A}\mathsf{A}^{\mathsf{T}}$$

$$\mathsf{K}_{\vec{Z}}^{-1} = (\mathsf{A}^{-1})^{\mathsf{T}}\mathsf{A}^{-1}$$

$$f_{\vec{Z}}(\vec{z}) = \frac{1}{(2\pi)^{k/2}\sqrt{\det(\mathsf{K}_{\vec{Z}})|}} \exp\left[-\frac{1}{2}\vec{z}^{\mathsf{T}}\mathsf{K}_{\vec{Z}}^{-1}\vec{z}\right]$$

For $\vec{Z} = Z_1, Z_2$, let $\mathbf{E}[Z_1^2] = \mathbf{K}_{11} = \sigma_1^2$, $\mathbf{E}[Z_2^2] = \mathbf{K}_{11} = \sigma_2^2$. Let ρ be normalized covariance

$$\rho = \frac{\mathbf{E}[Z_1 Z_2]}{\sigma_1 \sigma_2} = \frac{\mathbf{k}_{12}}{\sigma_1 \sigma_2}.$$

$$\det(\mathbf{K}_{\vec{Z}}) = \sigma_1^2 \sigma_2^2 - k_{12}^2 = \sigma_1^2 \sigma_2^2 (1 - \rho^2).$$

For A to be non-singular, we need $|\rho| < 1$. We then have

$$\mathbf{K}_{\vec{Z}^{-1}} == \frac{1}{1-\rho^2} \begin{bmatrix} 1/\sigma_1^2 & -\rho/(\sigma_1\sigma_2) \\ -\rho/(\sigma_1\sigma_2) & 1/\sigma_2^2 \end{bmatrix}$$

$$\vec{f}_{\vec{Z}}(\vec{z}) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \exp\left(\frac{-\frac{z_1}{\sigma_1}^2 + 2\rho\frac{z_1}{\sigma_1}\frac{z_2}{\sigma_2} - \frac{z_2}{\sigma_2}^2}{2(1-\rho^2)} \right)$$

Lesson: Even for $k = 2$, this is messy.

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