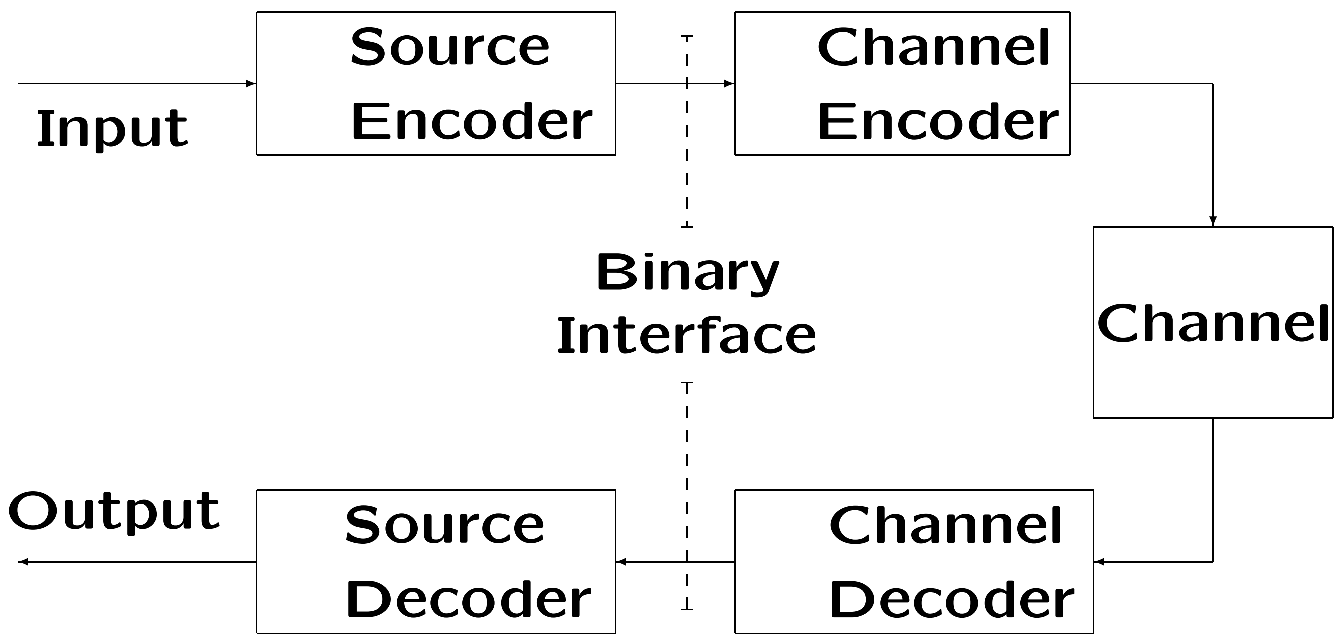
CHANNEL ENCODING & DECODING



Simplest Example of channel encoding

A sequence of binary digits is mapped, one at a time, into a sequence of signals from the constellation $\{1, -1\}$.

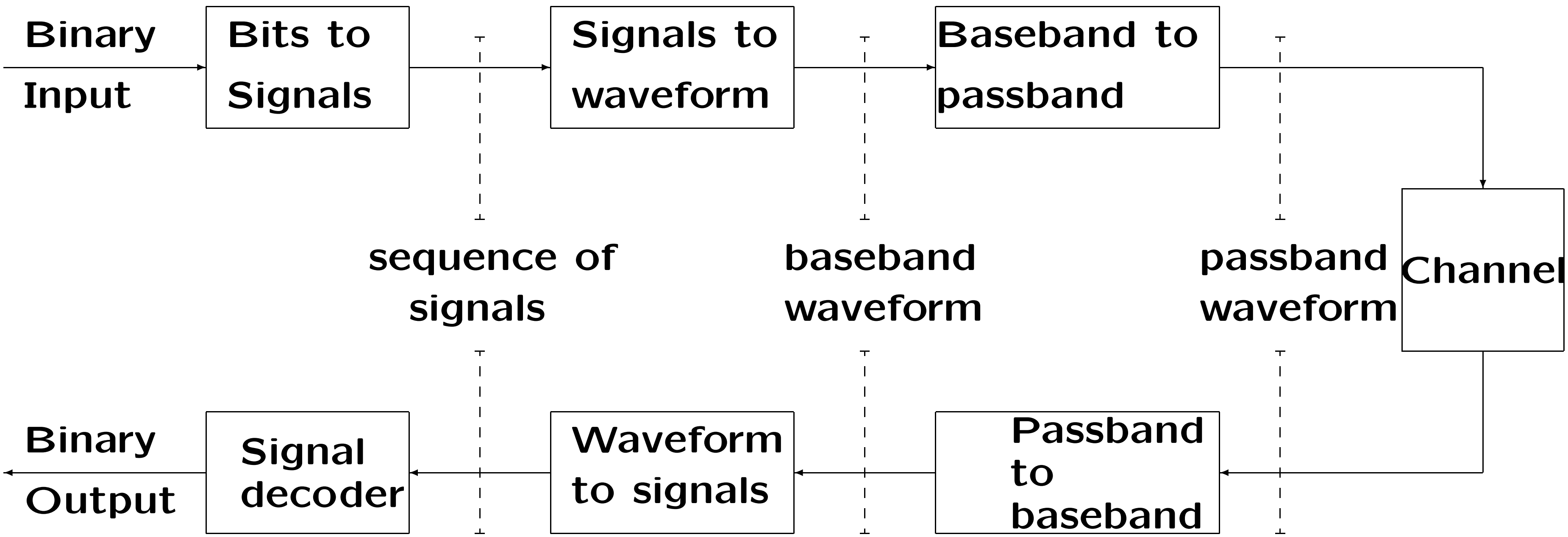
Usually the mapping is $0 \to 1$ and $1 \to -1$.

The sequence of signals, $u_1, u_2, \ldots$, is mapped to the waveform $\sum_k u_k \mathrm{sinc}\left(\frac{t}{T} - k\right)$.

With no noise, no delay, and no attenuation, the received waveform is $\sum_k u_k \mathrm{sinc}\left(\frac{t}{T} - k\right)$.

This is sampled and converted back to binary.

General structure



For now we focus on the baseband transmitted waveform $u(t)$ and the received baseband waveform $v(t)$.

The frequency translation from baseband to passband is undone at receiver.

Assume $u(t)$, the transmitted baseband waveform, equals the received waveform $v(t)$.

This ignores attenuation, delay, and noise.

Ignoring attenuation means amplitude scaling at receiver.

Attenuation is usually considered separately as part of the 'link budget.'

Assume $u(t)$, the transmitted baseband waveform, equals the received baseband waveform $v(t)$.

This ignores attenuation, delay, and noise.

This assumes time shifting at receiver. (Filters can be non-causal).

The time shifting is called 'timing recovery.' It locks the receiver clock to the transmitter clock plus propagation delay.

This means that filtering can be non-causal - it can be incorporated into the timing recovery.

Assume $u(t)$, the transmitted baseband waveform, equals the received baseband waveform $v(t)$.

This ignores attenuation, delay, and noise.

For now, we simply assume that noise can alter each received signal independently by at most a fixed amount.

This requires a minimum separation, say d, between the possible signals in a constellation.

We scale both received signal and noise so that $u(t)$ plus noise is received.

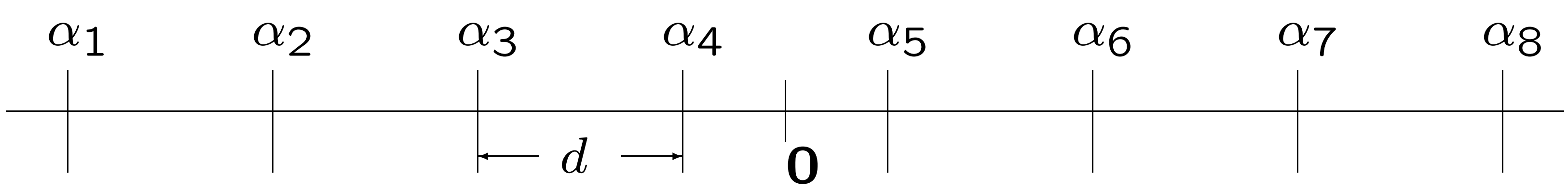
Pulse amplitude modulation (PAM)

The signals in PAM are one dimensional, i.e., the constellation is a set of $M = 2^b$ real numbers, where there are b binary inputs per signal.

It is modulated as $u(t) = \sum_k u_k p(t - kT)$, where $p(t)$ is the basic pulse shape.

A standard PAM signal set uses equi-spaced signals symmetric around 0.

$$\mathcal{A} = \{-d(M-1)/2, \dots, -d/2, d/2, \dots, d(M-1)/2\}.$$



8-PAM signal set

The signal energy, i.e., the mean square signal value assuming equiprobable signals, is

$$E_s = \frac{d^2(M^2 - 1)}{12} = \frac{d^2(2^{2b} - 1)}{12}.$$

This increases as d^2 and as M^2.

View noise as setting a minimum value for d.

Errors in reception are primarily due to noise exceeding $d/2$.

For many channels, the noise is independent of the signal, which explains the standard equal spacing between signal constellation values.

PAM Modulation

$$\{u_1, u_2, \dots\} \quad \rightarrow \quad u(t) = \sum_k u_k p(t - kT).$$

Modulation defined by interval T and basic waveform (pulse) $p(t)$.

$p(t)$ can be non-realizable ($p(t) \neq 0$ for $t < 0$), and could be $\text{sinc}(t/T)$.

This constrains waveform to baseband with limit $1/(2T)$.

$\text{sinc}(t/T)$ dies out impractically slowly with time; it also requires infinite delay at the transmitter.

We need a compromise between time decay and bandwidth.

We also would like to retrieve the coefficients u_k perfectly from $u(t)$ (assuming no noise).

Assume that the receiver filters $u(t)$ with an LTI filter with impulse response $q(t)$.

The filtered waveform $r(t) = \int u(\tau)q(\tau - t)\, d\tau$ is then sampled $r(0), r(T), \ldots$

The question is how to choose $p(t)$ and $q(t)$ so that $r(kT) = u_k$.

The question seems artificial (why choose a linear filter followed by sampling?)

We find later, when noise is added, that this all makes sense as a layered solution.

$$\begin{aligned} r(t) &= \int u(\tau)q(\tau - t)\, d\tau = \int_{-\infty}^{\infty} \sum_k u_k p(\tau - kT) q(t-\tau)\, d\tau. \\ &= \sum_k u_k g(t-kT) \quad \textbf{where} \quad g(t) = p(t) * q(t). \end{aligned}$$

Think of an impulse train $\sum_k u_k \delta(t-kT)$ passed through $p(t)$ and then $q(t)$.

While ignoring noise, $r(t)$ is determined by $g(t)$; $p(t)$ and $q(t)$ are otherwise irrelevant.

Definition: A waveform $g(t)$ is ideal Nyquist with period T if $g(kT) = \delta(k)$.

If $g(t)$ is ideal Nyquist, then $r(kT) = u_k$ for all $k \in \mathbb{Z}$. If $g(t)$ is not ideal Nyquist, then $r(kT) \neq u_k$ for some k and choice of $\{u_k\}$.

An ideal Nyquist $g(t)$ implies no <u>intersymbol interference</u> at the above receiver.

We will see that choosing $g(t)$ to be ideal Nyquist fits in nicely when looking at the real problem, which is coping with both noise and intersymbol interference.

$g(t) = \text{sinc}(t/T)$ **is ideal Nyquist. but has too much delay.**

If $g(t)$ is to be strictly baseband limited to $1/(2T)$**,** $\text{sinc}(t/T)$ **turns out to be the only solution.**

We look for compromise between bandwidth and delay.

Since ideal Nyquist is all about samples of $g(t)$, we look at aliasing again. The baseband reconstruction $s(t)$ from $\{g(kT)\}$ is

$$s(t) = \sum_k g(kT)\text{sinc}(\frac{t}{T} - k).$$

$g(t)$ **is ideal Nyquist iff** $s(t) = \text{sinc}(t/T)$ **i.e., iff**

$$\hat{s}(f) = T\,\text{rect}(fT)$$

From the aliasing theorem,

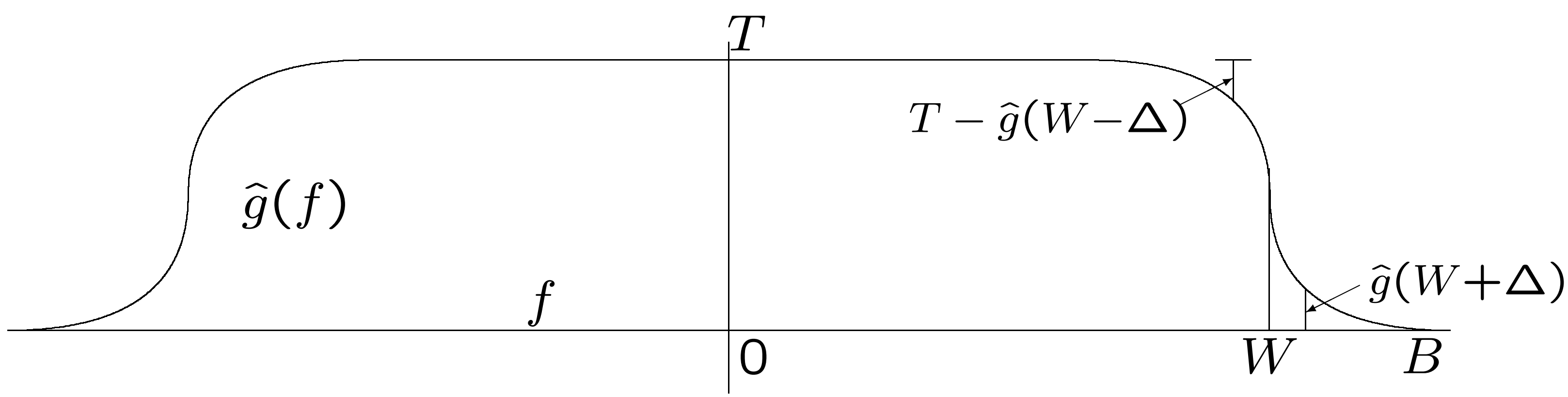
$$\hat{s}(f) = \sum_m \hat{g}(f + \frac{m}{T})\,\text{rect}(fT).$$

Thus $g(t)$ **is ideal Nyquist iff**

$$\sum_m \hat{g}(f + m/T)\,\text{rect}(fT) = T\,\text{rect}(fT)$$

This says that out of band frequencies can help in avoiding intersymbol interference.

We want to keep $\hat{g}(f)$ almost baseband limited to $1/(2T)$, and thus assume actual bandwidth B less than $1/T$.



This is a band edge symmetry requirement.

PAM filters in practice often have raised cosine transforms. The raised cosine frequency function, for any given rolloff α between 0 and 1, is defined by

$$\hat{g}_\alpha(f) = \begin{cases} T, & 0 \le |f| \le \frac{1-\alpha}{2T}; \\ T \ \cos^2\left[\frac{\pi T}{2\alpha}(|f| - \frac{1-\alpha}{2T})\right], & \frac{1-\alpha}{2T} \le |f| \le \frac{1+\alpha}{2T}; \\ 0, & |f| \ge \frac{1+\alpha}{2T}. \end{cases}$$

$$g_\alpha(t) = \text{sinc}(\frac{t}{T}) \ \frac{\cos(\pi\alpha t/T)}{1 - 4\alpha^2 t^2/T^2}$$

$\Re\{\hat{g}(f)\}$ **must satisfy the band edge symmetry condition to meet the Nyquist criterion.**

Choosing $\Im\{\hat{g}(f)\} \neq 0$ **simply increases the energy outside of the Nyquist band with little effect on delay.**

Thus we restrict $\hat{g}(f)$ **to be real (as in the raised cosine pulses used in practice).**

Because of noise, we choose $|\hat{p}(f)| = |\hat{q}(f)|$**.**

Since $\hat{g}(f) = \hat{p}(f)\hat{q}(f)$**, this requires** $\hat{q}(f) = \hat{p}^*(f)$ **and thus** $q(t) = p^*(-t)$**. This means that**

$$g(t) = \int p(\tau)q(t-\tau)\,d\tau = \int p(\tau)p^*(\tau - t)\,d\tau$$

For $g(t)$ ideal Nyquist,

$$g(kT) = \int p(\tau)p^*(\tau - kT)\,d\tau = \begin{cases} 1 & \textbf{for} \quad k = 0 \\ 0 & \textbf{for} \quad k \neq 0 \end{cases}$$

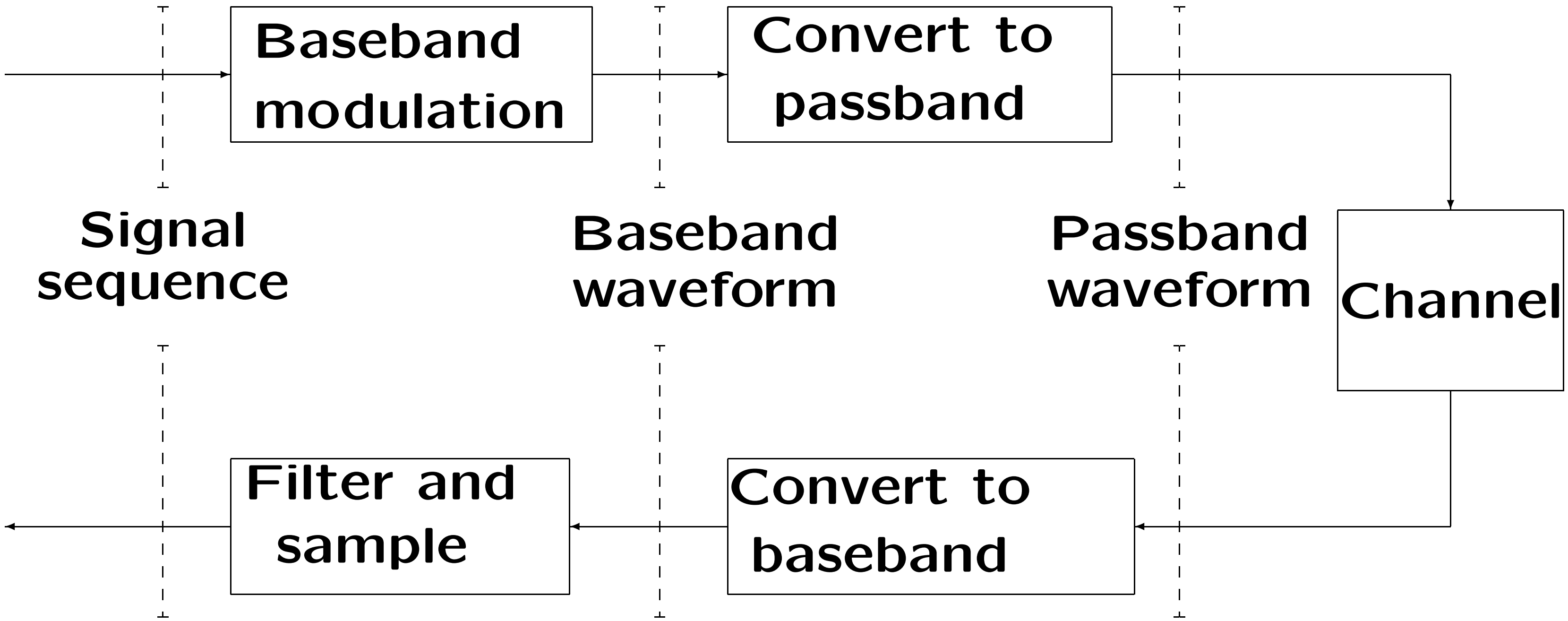
This means that $\{p(t-kT);\ k \in Z\}$ is an orthogonal set of functions.

These functions are all real $\mathcal{L}_2$ functions for PAM, but we allow the possibility of complex functions for later.

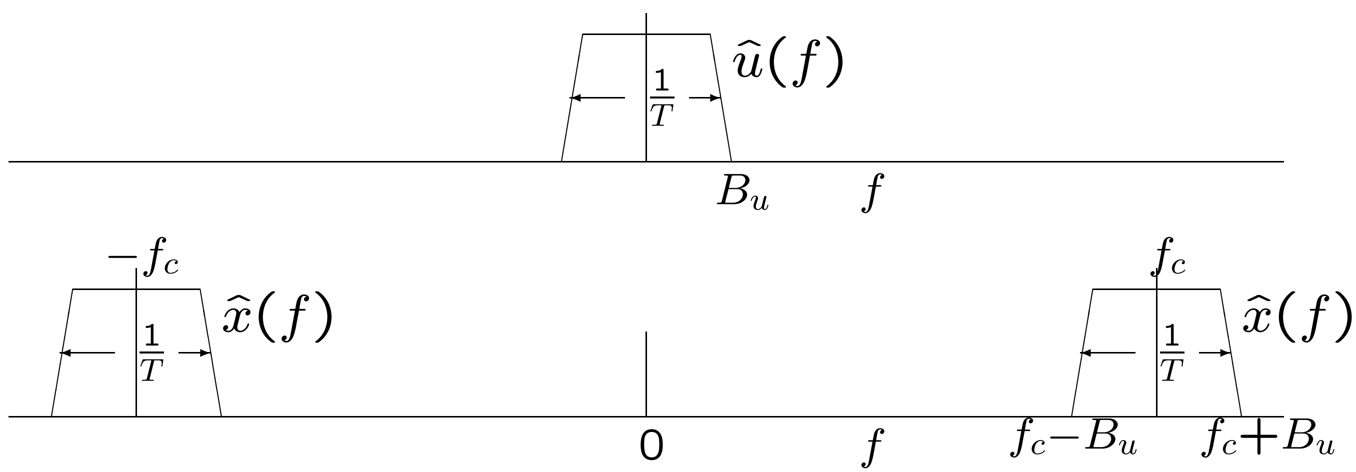
Since $|\widehat{p}(f)|^2 = \widehat{g}(f)$, $p(t)$ is often called square root of Nyquist.

In vector terms, $\int u(\tau)q(kT-\tau)\,d\tau$ is the projection of u on $p(t-kT)$. $q(t)$ is called the matched filter to $p(t)$.

FREQUENCY TRANSLATION



$$
\begin{aligned}
x(t) &= u(t)[e^{2\pi i f_c t} + e^{-2\pi i f_c t}] = 2u(t)\cos(2\pi f_c t), \\
\hat{x}(f) &= \hat{u}(f - f_c) + \hat{u}(f + f_c).
\end{aligned}
$$



The bandwidth B is $2B_u$. The bandwidth is always the range of positive frequencies used.

When a baseband waveform limited to B is shifted up to passband, the passband waveform becomes limited to $2B$.

For $u(t)$ real, $\hat{u}(f)$; $f \geq 0$ specifies $u(t)$.

Then $\hat{x}(f)$; $f_c \leq f \leq f_c + W_B$ specifies $u(t)$.

If $f_c - B_u \leq f \leq f_c$ filtered out of $x(t)$, result is single sideband (same for lower sideband).

Without filter, it is double sideband PAM.

SSB rare for data. DSB PAM common where frequency utilization is no problem.

QUADRATURE AMP. MOD. (QAM)

QAM solves the frequency waste problem of DSB AM by using a complex baseband waveform $u(t)$.

$$x(t) = u(t)e^{2\pi i f_c t} + u^*(t)e^{-2\pi i f_c t}\,.$$

DSB PAM is special case where $u(t)$ is real.

$$\begin{aligned} x(t) &= 2\Re\{u(t)e^{2\pi i f_c t}\} \\ &= 2\Re\{u(t)\}\cos(2\pi f_c t) - 2\Im\{u(t)\}\sin(2\pi f_c t)\,. \end{aligned}$$

It sends one baseband waveform on cos carrier, another on sine carrier.

Conceptually, QAM shifts complex $u(t)$ up by f_c. Then complex conjugate added to form real $x(t)$.

We think of the shifting and conjugating separately.

Binary $\Longrightarrow$ symbols $\Longrightarrow$ complex signals $\Longrightarrow$ $\Longrightarrow$ $u(t)$ $\Longrightarrow$ $u(t)e^{2\pi i f_c t}$ $\Longrightarrow$ $x(t)$.

At the receiver,

$x(t)$ $\Longrightarrow$ $u(t)e^{2\pi i f_c t}$ $\Longrightarrow$ $u(t)$ $\Longrightarrow$ $\Longrightarrow$ complex signals $\Longrightarrow$ symbols $\Longrightarrow$ Binary.

COMPLEX (QAM) SIGNAL SET

$R =$ in bits per second.

Segment b bits at a time ($M = 2^b$).

Map M symbols (binary b-tuples) to signal set.

Signal rate is $R_s = R/b$ signals per second.

$T = 1/R_s$ is the signal interval.

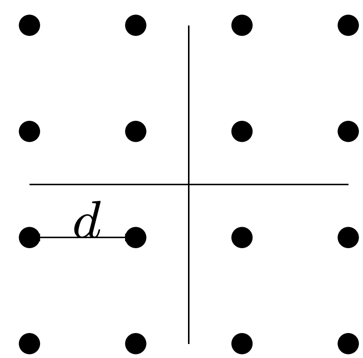
Signals $\{u_k\}$ are complex numbers (or real 2-tuples).

Signal set $\mathcal{A}$ is constellation of M complex numbers (or real 2-tuples)

A standard ($\sqrt{M} \times \sqrt{M}$)-QAM signal set is the Cartesian product of two $\sqrt{M}$-PAM sets; i.e.,

$$\mathcal{A} = \{(a' + ia'') \mid a' \in \mathcal{A}', a'' \in \mathcal{A}'\},$$

It is a square array of signal points located as below for $M = 16$.



The energy per 2D signal is

$$E_s = \frac{d^2[\sqrt{M}^2 - 1]}{6} = \frac{d^2[M - 1]}{6}.$$

Choosing a good signal set is similar to choosing a 2D set of representation points in quantization.

Here one essentially wants to choose M points all at distance at least d so as to minimize the energy of the signal set.

This is even uglier than quantization. Try to choose the optimal set of 8 signals with $d = 1$.

For the most part, standard signal sets are used.

SIGNALS TO COMPLEX WAVEFORM

Nyquist theory works without change (except modulation pulse can be complex).

Bandedge symmetry requires real $g(t)$.

More important, orthogonality of $p(t-kT)$ requires $g(t)$ to be real, but not $p(t)$.

Nominal passband Nyquist bandwidth is $1/T$. Actual passband bandwidth is 5 or 10 percent more.

BASEBAND TO PASSBAND

Assume that $B_u = B/2 < f_c$.

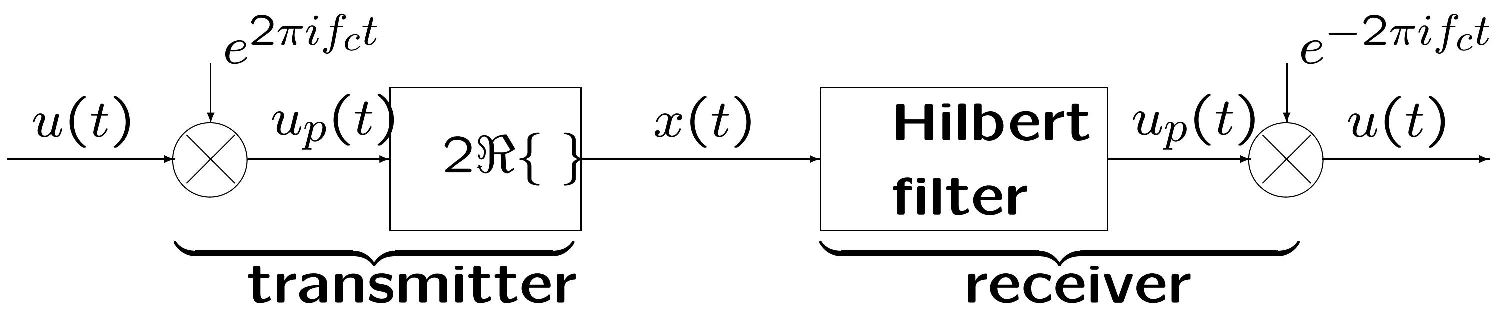
This ensures that $u(t)e^{2\pi i f_c t}$ is strictly in the positive frequency band.

It ensures that $\hat{u}(f + f_c)$ and $\hat{u}(f - f_c)$ do not overlap.

View as two step: $u_p(t) = u(t)e^{2\pi i f_c t}$. Then $x(t) = u_p(t) + u_p^*(t)$.

Note that $u_p(t)$ can be retrieved from $x(t)$ by a complex filter of frequency response $\hat{h}(f) = 1$ for $f > 0$, $\hat{h}(f) = 0$ for $f \leq 0$.

This is called a Hilbert filter.



Note that $u(t)$ is complex, and viewed as vector in complex $\mathcal{L}_2$.

$x(t)$ is real and viewed as vector in real $\mathcal{L}_2$.

Orthogonal expansions must be treated with great care.

This is nice for analysis, but not usually so for implementation.

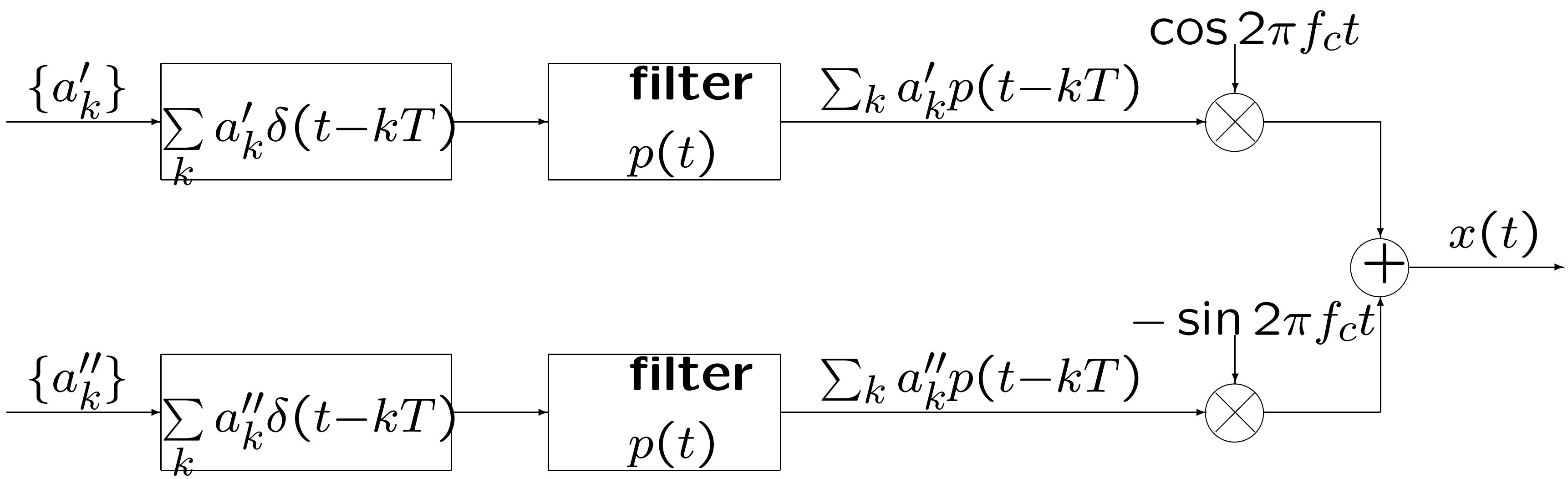
QAM IMPLEMENTATION (DSB-QC)

Assume $p(t)$ is real

$$\Re\{u(t)\} = \sum_k \Re\{u_k\}\, p(\frac{t}{T}-k),$$
$$\Im\{u(t)\} = \sum_k \Im\{u_k\}\, p(\frac{t}{T}-k).$$

With $u'_k = \Re\{u_k\}$ and $u''_k = \Im\{u_k\}$,

$$x(t) = 2\cos(2\pi f_c t)\left(\sum_k u'_k p(t-kT)\right) - 2\sin(2\pi f_c t)\left(\sum_k u''_k p(t-kT)\right)$$



Demodulate by multiplying $x(t)$ by both cosine and sign. Then filter out components arount $2f_c$.

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