

Inner product spaces

A vector space in itself contains no sense of distance or angles.

An inner product space is a vector space with an additional scalar valued operation $\langle \mathbf{v}, \mathbf{u} \rangle$ called an inner product and satisfying the following:

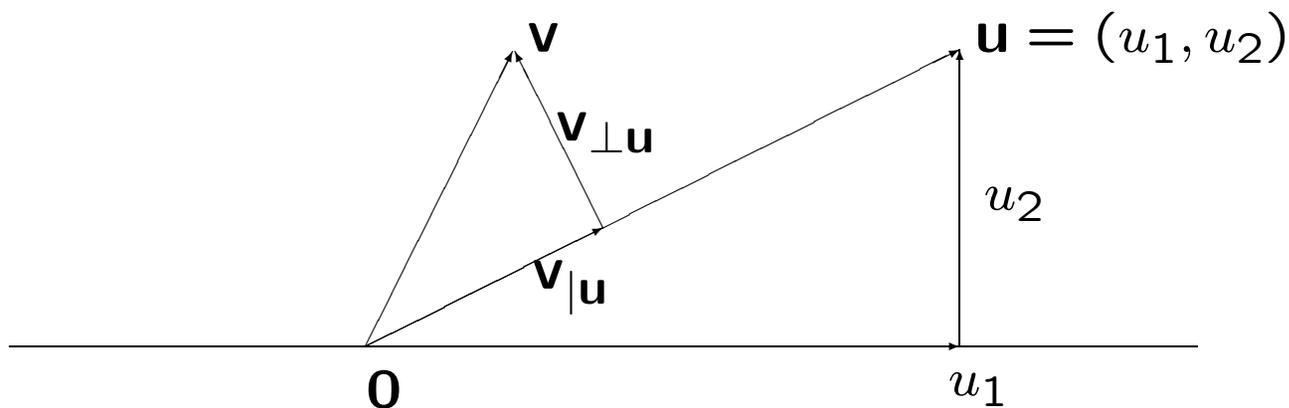
- 1. $\langle \mathbf{v}, \mathbf{u} \rangle = \langle \mathbf{u}, \mathbf{v} \rangle^*$**
- 2. $\langle \alpha \mathbf{v} + \beta \mathbf{u}, \mathbf{w} \rangle = \alpha \langle \mathbf{v}, \mathbf{w} \rangle + \beta \langle \mathbf{u}, \mathbf{w} \rangle$**
- 3. $\langle \mathbf{v}, \mathbf{v} \rangle \geq 0$, equality iff $\mathbf{v} = \mathbf{0}$.**

For \mathbb{R}^n or \mathbb{C}^n , we usually define $\langle \mathbf{v}, \mathbf{u} \rangle = \sum_i v_i u_i^*$.

In terms of unit vectors, $\langle \mathbf{v}, \mathbf{e}_i \rangle = v_i$, $\langle \mathbf{e}_i, \mathbf{v} \rangle = v_i^*$ and thus $\langle \mathbf{e}_i, \mathbf{e}_j \rangle = 0$ for $i \neq j$.

Definitions: $\|\mathbf{v}\|^2 = \langle \mathbf{v}, \mathbf{v} \rangle$ is squared norm of \mathbf{v} .
 $\|\mathbf{v}\|$ is length of \mathbf{v} . \mathbf{v} and \mathbf{u} are orthogonal if $\langle \mathbf{v}, \mathbf{u} \rangle = 0$.

More generally \mathbf{v} can be broken into a part $\mathbf{v}_{\perp \mathbf{u}}$ that is orthogonal to \mathbf{u} and another part $\mathbf{v}_{|\mathbf{u}}$ (the projection of \mathbf{v} on \mathbf{u}) that is collinear with \mathbf{u} .

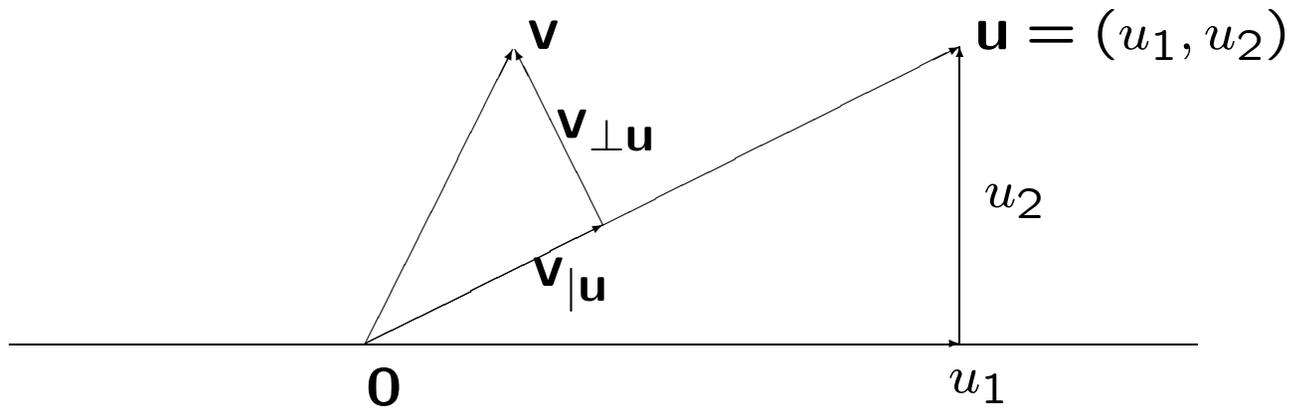


Theorem: (1D Projection) Let \mathbf{v} and $\mathbf{u} \neq \mathbf{0}$ be arbitrary vectors in a real or complex inner product space. Then there is a unique scalar α for which $\langle \mathbf{v} - \alpha\mathbf{u}, \mathbf{u} \rangle = 0$. That α is given by $\alpha = \langle \mathbf{v}, \mathbf{u} \rangle / \|\mathbf{u}\|^2$.

Proof: Calculate $\langle \mathbf{v} - \alpha\mathbf{u}, \mathbf{u} \rangle$ for an arbitrary scalar α and find the conditions under which it is zero:

$$\langle \mathbf{v} - \alpha\mathbf{u}, \mathbf{u} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle - \alpha\langle \mathbf{u}, \mathbf{u} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle - \alpha\|\mathbf{u}\|^2,$$

which is equal to zero if and only if $\alpha = \langle \mathbf{v}, \mathbf{u} \rangle / \|\mathbf{u}\|^2$.



$$\mathbf{v}|_{\mathbf{u}} = \frac{\langle \mathbf{v}, \mathbf{u} \rangle}{\|\mathbf{u}\|^2} \mathbf{u} = \left\langle \mathbf{v}, \frac{\mathbf{u}}{\|\mathbf{u}\|} \right\rangle \frac{\mathbf{u}}{\|\mathbf{u}\|}$$

$$\frac{\mathbf{v}|_{\mathbf{u}}}{\|\mathbf{v}\|} = \left\langle \frac{\mathbf{v}}{\|\mathbf{v}\|}, \frac{\mathbf{u}}{\|\mathbf{u}\|} \right\rangle \frac{\mathbf{u}}{\|\mathbf{u}\|}$$

$$\cos(\angle(\mathbf{u}, \mathbf{v})) = \left\langle \frac{\mathbf{v}}{\|\mathbf{v}\|}, \frac{\mathbf{u}}{\|\mathbf{u}\|} \right\rangle$$

Pythagorean theorem: For \mathbf{v}, \mathbf{u} orthogonal,

$$\|\mathbf{u} + \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$$

Proof: $\langle \mathbf{u} + \mathbf{v}, \mathbf{u} + \mathbf{v} \rangle = \|\mathbf{u}\|^2 + \langle \mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{v}, \mathbf{u} \rangle + \|\mathbf{v}\|^2$

For projection,

$$\|\mathbf{v}\|^2 = \|\mathbf{v}_{|\mathbf{u}}\|^2 + \|\mathbf{v}_{\perp \mathbf{u}}\|^2$$

It follows that

$$\|\mathbf{v}\|^2 \geq \|\mathbf{v}_{|\mathbf{u}}\|^2 = \frac{|\langle \mathbf{v}, \mathbf{u} \rangle|^2}{\|\mathbf{u}\|^4} \|\mathbf{u}\|^2$$

This yields the Schwartz inequality,

$$|\langle \mathbf{u}, \mathbf{v} \rangle| \leq \|\mathbf{u}\| \|\mathbf{v}\|$$

In normalized form, $|\langle \frac{\mathbf{u}}{\|\mathbf{u}\|}, \frac{\mathbf{v}}{\|\mathbf{v}\|} \rangle| \leq 1$

If we define the inner product of \mathcal{L}_2 as

$$\langle \mathbf{u}, \mathbf{v} \rangle = \int_{-\infty}^{\infty} u(t)v^*(t)dt,$$

then \mathcal{L}_2 becomes an inner product space.

Because $\langle \mathbf{u}, \mathbf{u} \rangle \neq 0$ for $\mathbf{u} \neq 0$, we must define equality as \mathcal{L}_2 equivalence.

The vectors in this space are equivalence classes.

Alternatively, view a vector as a set of coefficients in an orthogonal expansion.

VECTOR SUBSPACES

A subspace of a vector space \mathcal{V} is a subset \mathcal{S} of \mathcal{V} that forms a vector space in its own right.

Equivalent: For all $u, v \in \mathcal{S}$, $\alpha u + \beta v \in \mathcal{S}$

Important: \mathbb{R}^n is not a subspace of \mathbb{C}^n ; real \mathcal{L}_2 is not a subspace of complex \mathcal{L}_2 .

A subspace of an inner product space (using the same inner product) is an inner product space.

DIMENSION (of \mathcal{V} or subspace)

The vectors $\mathbf{v}_1, \dots, \mathbf{v}_n \in \mathcal{V}$ span \mathcal{V} if every vector \mathbf{u} in \mathcal{V} is a linear combination, $\mathbf{u} = \sum_{i=1}^n \alpha_i \mathbf{v}_i$.

\mathcal{V} is finite dimensional if it is spanned by a finite set of vectors.

The vectors $\mathbf{v}_1, \dots, \mathbf{v}_n \in \mathcal{V}$ are linearly independent if $\mathbf{u} = \sum_{i=1}^n \alpha_i \mathbf{v}_i = \mathbf{0}$ only for $\alpha_i = 0, 1 \leq i \leq n$.

The vectors $\mathbf{v}_1, \dots, \mathbf{v}_n \in \mathcal{V}$ are a basis for \mathcal{V} if they are lin. ind. and span \mathcal{V} .

Theorem: If $\mathbf{v}_1, \dots, \mathbf{v}_n$ span \mathcal{V} , then a subset is a basis of \mathcal{V} . If \mathcal{V} is finite dim., then every basis has the same size, and any lin. ind. set $\mathbf{v}_1, \dots, \mathbf{v}_n$ is part of a basis .

If \mathcal{V} is an inner product space and \mathcal{S} is a subspace, then \mathcal{S} is an inner product space with that inner product.

Assume \mathcal{V} is an inner product space in what follows.

A vector $\phi \in \mathcal{V}$ is normalized if $\|\phi\| = 1$.

The projection $\mathbf{v}|_{\phi} = \langle \mathbf{u}, \phi \rangle \phi$ for $\|\phi\| = 1$

An orthonormal set $\{\phi_j\}$ is a set such that $\langle \phi_j, \phi_k \rangle = \delta_{jk}$

If $\{\mathbf{v}_j\}$ is orthogonal set, then $\{\phi_j\}$ is an orthonormal set where $\phi_j = \mathbf{v}_j / \|\mathbf{v}_j\|$.

Projection theorem: Assume that $\{\phi_1, \dots, \phi_n\}$ is an orthonormal basis for an n -dimensional subspace $\mathcal{S} \subset \mathcal{V}$. For each $\mathbf{v} \in \mathcal{V}$, there is a unique $\mathbf{v}_{|\mathcal{S}} \in \mathcal{S}$ such that $\langle \mathbf{v} - \mathbf{v}_{|\mathcal{S}}, \mathbf{s} \rangle = 0$ for all $\mathbf{s} \in \mathcal{S}$. Furthermore,

$$\mathbf{v}_{|\mathcal{S}} = \sum_j \langle \mathbf{v}, \phi_j \rangle \phi_j.$$

Proof outline: Let $\mathbf{v}_{|\mathcal{S}} = \sum_i \alpha_i \phi_i$. Find the conditions on $\alpha_1, \dots, \alpha_n$ such that $\mathbf{v} - \mathbf{v}_{|\mathcal{S}}$ is orthogonal to each ϕ_i .

$$0 = \langle \mathbf{v} - \sum_i \alpha_i \phi_i, \phi_j \rangle = \langle \mathbf{v}, \phi_j \rangle - \alpha_j$$

Thus $\alpha_j = \langle \mathbf{v}, \phi_j \rangle$ and $\mathbf{v}_{|\mathcal{S}} = \sum_j \langle \mathbf{v}, \phi_j \rangle \phi_j$.

For $\mathbf{v} \in \mathcal{S}$, $\mathbf{v} = \sum_j \alpha_j \phi_j$, $\{\phi_j\}$ orthonormal basis of \mathcal{S} ,

$$\|\mathbf{v}\|^2 = \langle \mathbf{v}, \sum_j \alpha_j \phi_j \rangle = \sum_j \alpha_j^* \langle \mathbf{v}, \phi_j \rangle = \sum_j |\alpha_j|^2$$

For arbitrary $\mathbf{v} \in \mathcal{V}$,

$$\|\mathbf{v}\|^2 = \|\mathbf{v}_{|\mathcal{S}}\|^2 + \|\mathbf{v}_{\perp\mathcal{S}}\|^2 \quad \text{(Pythagoras)}$$

$$0 \leq \|\mathbf{v}_{|\mathcal{S}}\|^2 \leq \|\mathbf{v}\|^2 \quad \text{(Norm bounds)}$$

$$0 \leq \sum_{j=1}^n |\langle \mathbf{v}, \phi_j \rangle|^2 \leq \|\mathbf{v}\|^2 \quad \text{(Bessel's inequality).}$$

$$\|\mathbf{v} - \mathbf{v}_{|\mathcal{S}}\| \leq \|\mathbf{v} - \mathbf{s}\| \quad \text{for any } \mathbf{s} \in \mathcal{S} \quad \text{(LS property).}$$

Gram-Schmidt orthonormalization

Given basis $\mathbf{s}_1, \dots, \mathbf{s}_n$ for an inner product subspace, find an orthonormal basis.

$\phi_1 = \mathbf{s}_1 / \|\mathbf{s}_1\|$ is an orthonormal basis for subspace \mathcal{S}_1 generated by \mathbf{s}_1 .

Given orthonormal basis ϕ_1, \dots, ϕ_k of subspace \mathcal{S}_k generated by $\mathbf{s}_1, \dots, \mathbf{s}_k$, project \mathbf{s}_{k+1} onto \mathcal{S}_k .

$$\phi_{k+1} = \frac{(\mathbf{s}_{k+1})_{\perp \mathcal{S}_k}}{\|(\mathbf{s}_{k+1})_{\perp \mathcal{S}_k}\|}$$

For \mathcal{L}_2 , the projection theorem can be extended to a countably infinite dimension.

Given any orthogonal set of functions θ_i , we can generate orthonormal functions as $\phi_i = \theta_i / \|\theta_i\|$.

Theorem: Let $\{\phi_m, 1 \leq m < \infty\}$ be a set of orthonormal functions, and let \mathbf{v} be any \mathcal{L}_2 vector. Then there is a unique \mathcal{L}_2 vector \mathbf{u} such that $\mathbf{v} - \mathbf{u}$ is orthogonal to each ϕ_n and

$$\lim_{n \rightarrow \infty} \left\| \mathbf{u} - \sum_{m=1}^n \langle \mathbf{v}, \phi_m \rangle \phi_m \right\| = 0.$$

$$\lim_{n \rightarrow \infty} \left\| \mathbf{u} - \sum_{m=1}^n \langle \mathbf{v}, \phi_m \rangle \phi_m \right\| = 0; \quad \langle \mathbf{v} - \mathbf{u}, \phi_j \rangle = 0 \quad \text{for all } j.$$

Outline of proof

Let \mathcal{S}_n be subspace spanned by ϕ_1, \dots, ϕ_n .

$$\mathbf{v}|_{\mathcal{S}_n} = \sum_{k=1}^n \alpha_k \phi_k, \quad \alpha_k = \langle \mathbf{v}, \phi_k \rangle$$

$$\|\mathbf{v}|_{\mathcal{S}_m} - \mathbf{v}|_{\mathcal{S}_n}\|^2 = \sum_{k=n}^m |\alpha_k|^2 \rightarrow 0$$

$\mathbf{v}|_{\mathcal{S}_n}$ forms a Cauchy sequence. By the Riesz-Fischer theorem, l.i.m. $\mathbf{v}|_{\mathcal{S}_n} = \mathbf{u}$ exists.

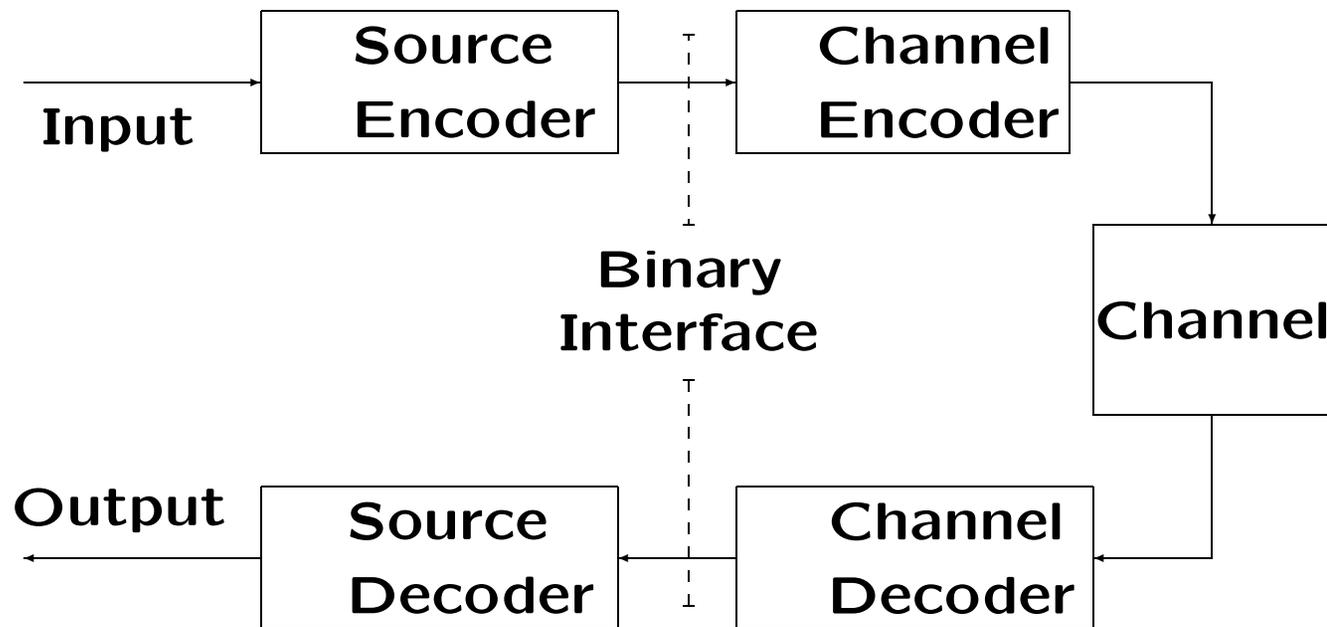
This shows that the fourier series converges in \mathcal{L}_2 .

Question: does $\sum_k \hat{v}_k e^{2\pi ikt/T}$ converge to $v(t)$ for \mathcal{L}_2 function $\{v(t); [-T/2, T/2] \rightarrow \mathbb{C}\}$.

Answer: Yes. In other words, $\{T^{-1/2} e^{2\pi ikt/T}\}$ spans the \mathcal{L}_2 functions on $[-T/2, T/2]$.

We also see that $v - v|_{\mathcal{S}_n}$ is orthogonal to v and is the LS approximation to v in \mathcal{S}_n .

CHANNEL ENCODING & DECODING



Simplest Example: A sequence of binary digits is mapped into a sequence of signals from the constellation $\{1, -1\}$.

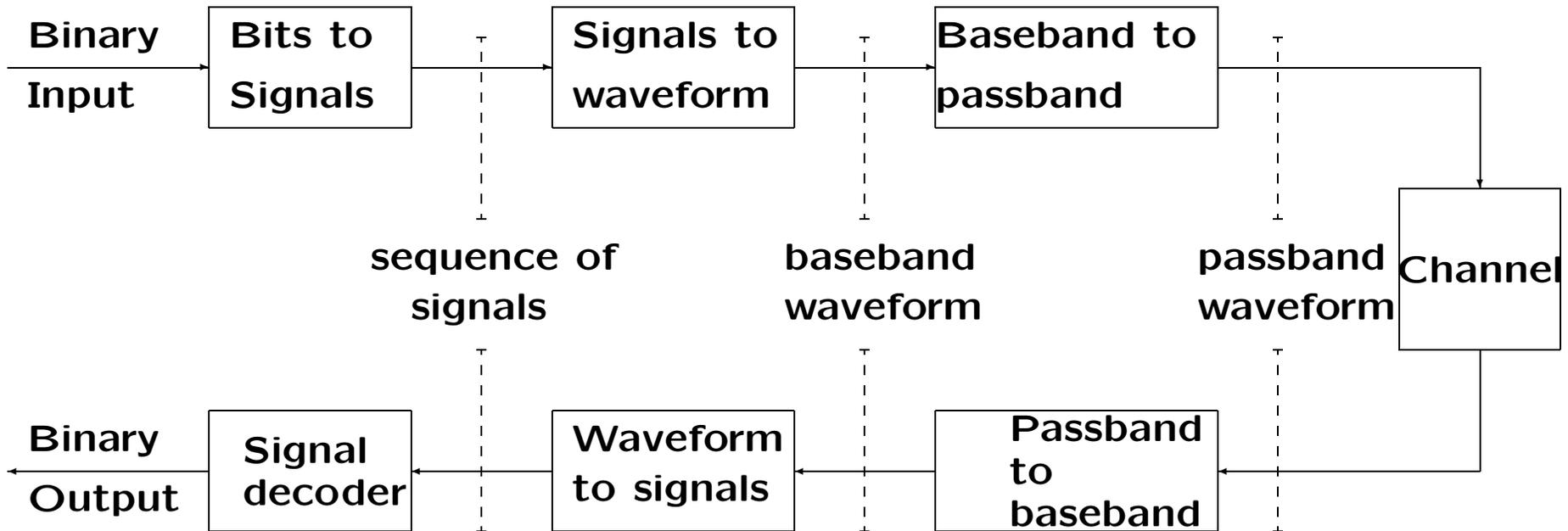
Usually the mapping is $0 \rightarrow 1$ and $1 \rightarrow -1$.

The sequence of signals, u_1, u_2, \dots , is mapped to the waveform $\sum_k u_k \text{sinc}\left(\frac{t}{T} - k\right)$.

With no noise, no delay, and no attenuation, the received waveform is $\sum_k u_k \text{sinc}\left(\frac{t}{T} - k\right)$.

This is sampled and converted back to binary.

General structure



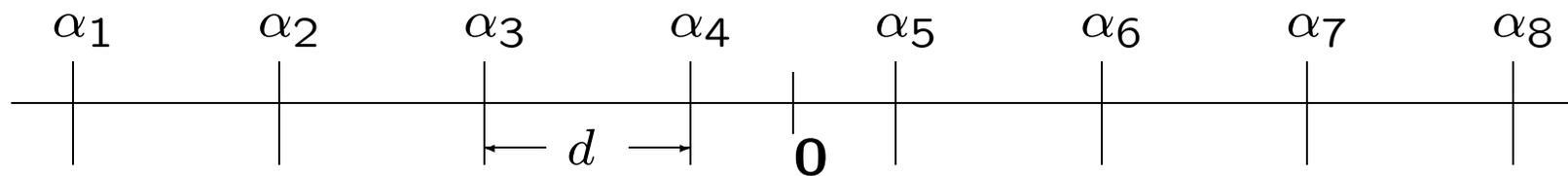
Pulse amplitude modulation (PAM)

The signals in PAM are one dimensional, i.e., the constellation is a set of real numbers.

It is modulated as $u(t) = \sum_k u_k p(t - kT)$.

A standard PAM signal set uses equi-spaced signals symmetric around 0.

$$\mathcal{A} = \{-d(M-1)/2, \dots, -d/2, d/2, \dots, d(M-1)/2\}.$$



8-PAM signal set

The signal energy, i.e., the mean square signal value assuming equiprobable signals, is

$$E_s = \frac{d^2(M^2 - 1)}{12} = \frac{d^2(2^{2b} - 1)}{12}.$$

This increases as d^2 and as M^2 .

We discuss noise later, but essentially the noise determines the allowable value of d .

Errors in reception are primarily due to noise exceeding $d/2$.

For many channels, the noise is independent of the signal, which explains the standard equal spacing between signal constellation values.

We usually assume that the received waveform is the same as the transmitted waveform.

That is, we ignore delay and attenuation.

Delay is ignored since ‘timing recovery’ at the receiver locks the receiver clock to the transmitter clock plus propagation delay.

The attenuation is usually considered separately as part of the ‘link budget.’

We scale both received signal and noise so that $u(t)$ plus noise is received.

PAM Modulation

$$\{u_1, u_2, \dots\} \rightarrow u(t) = \sum_k u_k p(t - kT).$$

Modulation defined by interval T and basic waveform (pulse) $p(t)$.

$p(t)$ can be non-realizable ($p(t) \neq 0$ for $t < 0$), and could be $\text{sinc}(t/T)$.

This constrains waveform to baseband with limit $1/(2T)$.

$\text{sinc}(t/T)$ dies out impractically slowly with time; it also requires infinite delay at the transmitter.

We need a compromise between time decay and bandwidth.

We also would like to retrieve the coefficients u_k perfectly from $u(t)$ (assuming no noise).

Assume that the receiver filters $u(t)$ with an LTI filter with impulse response $q(t)$.

The filtered waveform $r(t) = \int u(\tau)q(\tau - t) d\tau$ is then sampled $r(0), r(T), \dots$

The question is how to choose $p(t)$ and $q(t)$ so that $r(kT) = u_k$.

The question seems artificial (why choose a linear filter followed by sampling?)

We find later, when noise is added, that this all makes sense as a layered solution.

$$\begin{aligned}
r(t) &= \int u(\tau)q(\tau - t) d\tau = \int_{-\infty}^{\infty} \sum_k u_k p(\tau - kT)q(t - \tau) d\tau. \\
&= \sum_k u_k g(t - kT) \quad \text{where} \quad g(t) = p(t) * q(t).
\end{aligned}$$

Think of an impulse train $\sum_k u_k \delta(t - kT)$ passed through $p(t)$ and then $q(t)$.

While ignoring noise, $r(t)$ is determined by $g(t)$; $p(t)$ and $q(t)$ are otherwise irrelevant.

Definition: A waveform $g(t)$ is ideal Nyquist with period T if $g(kT) = \delta(k)$.

If $g(t)$ is ideal Nyquist, then $r(kT) = u_k$ for all $k \in \mathbb{Z}$. If $g(t)$ is not ideal Nyquist, then $r(kT) \neq u_k$ for some k and choice of $\{u_k\}$.

An ideal Nyquist $g(t)$ implies no intersymbol interference at the above receiver.

We will see that choosing $g(t)$ to be ideal Nyquist fits in nicely when looking at the real problem, which is coping with both noise and intersymbol interference.

$g(t) = \text{sinc}(t/T)$ is ideal Nyquist. but has too much delay.

If $g(t)$ is to be strictly baseband limited to $1/(2T)$, $\text{sinc}(t/T)$ turns out to be the only solution.

We look for compromise between bandwidth and delay.

Since ideal Nyquist is all about samples of $g(t)$, we look at aliasing again. The baseband reconstruction $s(t)$ from $\{g(kT)\}$ is

$$s(t) = \sum_k g(kT) \operatorname{sinc}\left(\frac{t}{T} - k\right).$$

$g(t)$ is ideal Nyquist iff $s(t) = \operatorname{sinc}(t/T)$ i.e., iff

$$\hat{s}(f) = T \operatorname{rect}(fT)$$

From the aliasing theorem,

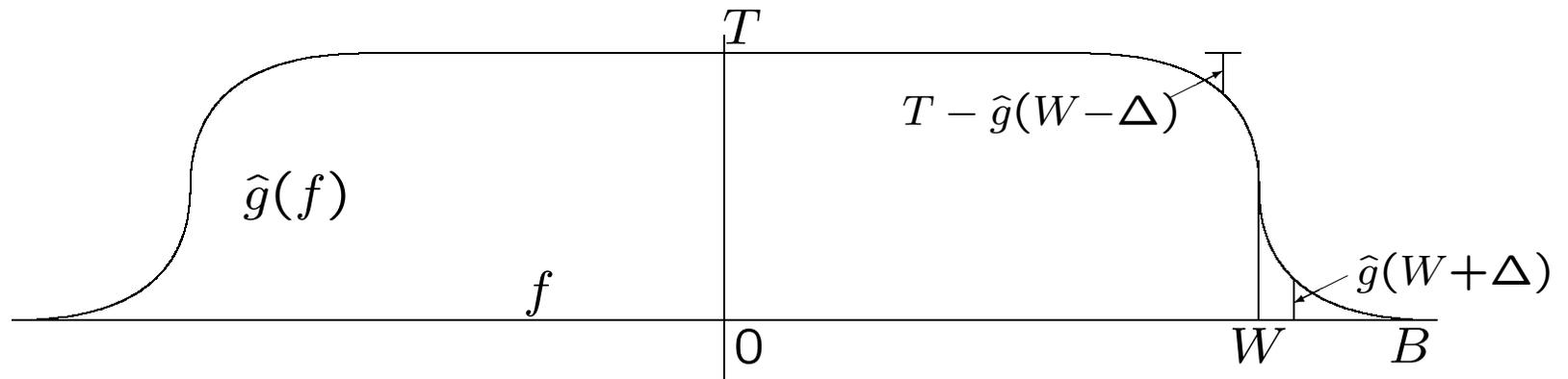
$$\hat{s}(f) = \sum_m \hat{g}\left(f + \frac{m}{T}\right) \operatorname{rect}(fT).$$

Thus $g(t)$ is ideal Nyquist iff

$$\sum_m \hat{g}\left(f + \frac{m}{T}\right) \operatorname{rect}(fT) = T \operatorname{rect}(fT)$$

This says that out of band frequencies can help in avoiding intersymbol interference.

We want to keep $\hat{g}(f)$ almost baseband limited to $1/(2T)$, and thus assume actual bandwidth B less than $1/T$.



This is a band edge symmetry requirement.

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