Functions not limited in time

We can segment an arbitrary $\mathcal{L}_2$ function into segments of width T. The mth segment is $u_m(t) = u(t)\text{rect}(t/T - m)$. We then have

$$u(t) = \text{l.i.m.}_{m_0\to\infty} \sum_{m=-m_0}^{m_0} u_m(t)$$

This works because $u(t)$ is $\mathcal{L}_2$. The energy in $u_m(t)$ must go to 0 as $m \to \infty$.

By shifting $u_m(t)$, we get the Fourier series:

$$u_m(t) \;=\; \text{l.i.m.} \sum_k \hat{u}_{k,m}\, e^{2\pi ikt/T}\, \text{rect}(\frac{t}{T} - m), \qquad \textbf{where}$$

$$\hat{u}_{k,m} \;=\; \frac{1}{T}\int_{-\infty}^{\infty} u(t)e^{-2\pi ikt/T}\text{rect}(\frac{t}{T} - m)\, dt, \;\; -\infty < k < \infty.$$

This breaks $u(t)$ into a double sum expansion of orthogonal functions, first over segments, then over frequencies.

$$u(t) = \text{l.i.m.} \sum_{m,k} \hat{u}_{k,m}\, e^{2\pi ikt/T}\, \text{rect}(\frac{t}{T} - m)$$

For each m, k and m', k', $e^{2\pi ikt/T}\, \text{rect}(\frac{t}{T} - m)$ is orthogonal to $e^{2\pi ik't/T}\, \text{rect}(\frac{t}{T} - m')$.

For $m \neq m'$, these functions are non-overlapping. For $m = m'$ and $k \neq k'$, they are orthogonal by Fourier series properties. This is the first of a number of orthogonal expansions of arbitrary $\mathcal{L}_2$ functions.

We call this the T-spaced truncated sinusoid expansion.

$$u(t) = \text{l.i.m.} \sum_{m,k} \hat{u}_{k,m} \, e^{2\pi ikt/T} \, \text{rect}(\frac{t}{T} - m)$$

This is the conceptual basis for algorithms such as voice compression that segment the waveform and then process each segment.

It matches our intuition about frequency well; that is, in music, notes (frequencies) keep changing.

The awkward thing is that the segmentation parameter T is arbitrary and not fundamental.

Fourier transform: $u(t) : \mathbb{R} \to \mathbb{C}$ **to** $\hat{u}(f) : \mathbb{R} \to \mathbb{C}$

$$\hat{u}(f) = \int_{-\infty}^{\infty} u(t)e^{-2\pi ift}\,dt.$$

$$u(t) = \int_{-\infty}^{\infty} \hat{u}(f)e^{2\pi ift}\,df.$$

For "well-behaved functions," first integral exists for all f**, second exists for all** t **and results in original** $u(t)$**.**

What does well-behaved mean? It means that the above is true.

$au(t) + bv(t)$	$\leftrightarrow$	$a\hat{u}(f) + b\hat{v}(f)$.	**Linearity**
$u^*(-t)$	$\leftrightarrow$	$\hat{u}^*(f)$.	**Conjugate**
$\hat{u}(t)$	$\leftrightarrow$	$u(-f)$.	**Duality**
$u(t-\tau)$	$\leftrightarrow$	$e^{-2\pi i f\tau}\hat{u}(f)$	**Time shift**
$u(t)e^{2\pi i f_0 t}$	$\leftrightarrow$	$\hat{u}(f-f_0)$	**Frequency shift**
$u(t/T)$	$\leftrightarrow$	$T\,\hat{u}(fT)$.	**Scaling**
$du(t)/dt$	$\leftrightarrow$	$i2\pi f\hat{u}(f)$.	**Differentiation**
$\int_{-\infty}^{\infty} u(\tau)v(t-\tau)\,d\tau$	$\leftrightarrow$	$\hat{u}(f)\hat{v}(f)$.	**Convolution**
$\int_{-\infty}^{\infty} u(\tau)v^*(\tau-t)\,d\tau$	$\leftrightarrow$	$\hat{u}(f)\hat{v}^*(f)$.	**Correlation**

Two useful special cases of any Fourier transform pair are:

$$u(0) = \int_{-\infty}^{\infty} \hat{u}(f)\, df;$$

$$\hat{u}(0) = \int_{-\infty}^{\infty} u(t)\, dt.$$

Parseval's theorem:

$$\int_{-\infty}^{\infty} u(t)v^*(t)\, dt = \int_{-\infty}^{\infty} \hat{u}(f)\hat{v}^*(f)\, df.$$

Replacing $v(t)$ by $u(t)$ yields the energy equation,

$$\int_{-\infty}^{\infty} |u(t)|^2\, dt = \int_{-\infty}^{\infty} |\hat{u}(f)|^2\, df.$$

Fourier transform: $u(t) : \mathbb{R} \to \mathbb{C}$ **to** $\hat{u}(f) : \mathbb{R} \to \mathbb{C}$

$$\hat{u}(f) = \int_{-\infty}^{\infty} u(t) e^{-2\pi i f t}\, dt.$$

$$u(t) = \int_{-\infty}^{\infty} \hat{u}(f) e^{2\pi i f t}\, df.$$

If $u(t)$ **is** $\mathcal{L}_1$**, first integral exists for all** f**. Furthermore** $\hat{u}(f)$ **must be a continuous function.**

If $\hat{u}(f)$ **is** $\mathcal{L}_1$**, second integral exists for all** t **and** $u(t)$ **is continuous.**

Unfortunately, we don't always get back to same function that we started with.

Not enough functions are $\mathcal{L}_1$ to provide suitable models for communication systems.

For example, $\text{sinc}(t)$ **is not $\mathcal{L}_1$.**

Also, functions with discontinuities cannot be Fourier transforms of $\mathcal{L}_1$ functions.

Finally, $\mathcal{L}_1$ functions might have infinite energy.

$\mathcal{L}_2$ functions turn out to be the "right" class.

$\{u(t) : \mathbb{R} \to \mathbb{C}\}$ **is $\mathcal{L}_2$ if measurable and if**

$$\int_{-\infty}^{\infty} |u(t)|^2 \, dt < \infty$$

$\mathcal{L}_2$ functions and Fourier transforms

Theorem: If $\{u(t) : \mathbb{R} \to \mathbb{C}\}$ is $\mathcal{L}_2$ and time limited, it is also $\mathcal{L}_1$.

Proof: $|u(t)|^2 \le |u(t)| + 1$, **so**

$$\int_{-A}^{B} |u(t)|\,dt \le \int_{-A}^{B} |u(t)|^2\,dt + (B - A) < \infty. \qquad \square$$

For any $\mathcal{L}_2$ function $u(t)$ and $A > 0$, let $v_A(t)$ be $u(t)$ truncated to $[-A, A]$,

$$v_A(t) = u(t)\text{rect}(\frac{t}{2A})$$

Then $v_A(t)$ is both $\mathcal{L}_2$ and $\mathcal{L}_1$. Its Fourier transform $\hat{v}_A(f)$ exists for all f and is continuous.

$$\hat{v}_A(f) = \int_{-A}^{A} u(t) e^{-2\pi i f t}\; dt.$$

$\mathcal{L}_2$ functions and Fourier transforms

Recall that if $\{u(t) : \mathbb{R} \to \mathbb{C}\}$ is $\mathcal{L}_2$ and time limited, it is also $\mathcal{L}_1$.

Proof: $|u(t)| \leq |u(t)|^2 + 1$, **so**

$$\int_{-A}^{B} |u(t)|\, dt \leq \int_{-A}^{B} |u(t)|^2\, dt + (B - A) < \infty. \quad \square$$

For any $\mathcal{L}_2$ function $u(t)$ and $A > 0$, consider $u(t)\text{rect}(\frac{t}{2A})$. **This function is time limited and $\mathcal{L}_2$, so it is $\mathcal{L}_1$.**

Its Fourier transform $\hat{v}_A(f)$ exists for all f and is continuous.

$$\hat{v}_A(f) = \int_{-A}^{A} u(t) e^{-2\pi i f t}\; dt.$$

Plancherel 1: There is an $\mathcal{L}_2$ function $\hat{u}(f)$ (the Fourier transform of $u(t)$), which satisfies the energy equation and

$$\lim_{A\to\infty} \int_{-\infty}^{\infty} |\hat{u}(f) - \hat{v}_A(f)|^2 \, dt = 0.$$

This is plausible since

$$\lim_{A\to\infty} \int |u(t) - v_A(t)|^2 \, dt = 0.$$

We denote this function $\hat{u}(f)$ as

$$\hat{u}(f) \;=\; \text{l.i.m.} \int_{-\infty}^{\infty} u(t) e^{2\pi i f t} \, dt.$$

Although $\{\hat{v}_A(f)\}$ is continuous for all $A \in \mathbb{R}$, $\hat{u}(f)$ is not necessarily continuous.

Similarly, for $B > 0$, consider the finite bandwidth approximation $\hat{u}(f)\text{rect}(\frac{f}{2B})$. This is $\mathcal{L}_1$ as well as $\mathcal{L}_2$,

$$u_B(t) = \int_{-B}^{B} \hat{u}(f) e^{2\pi i f t}\, df \qquad (1)$$

exists for all $t \in \mathbb{R}$ and is continuous.

Plancherel 2: For any $\mathcal{L}_2$ function $u(t)$, let $\hat{u}(f)$ be the FT of Plancherel 1. Then

$$\lim_{B\to\infty} \int_{-\infty}^{\infty} |u(t) - w_B(t)|^2 \, dt = 0. \qquad (2)$$

$$u(t) = \text{l.i.m.} \int_{-\infty}^{\infty} \hat{u}(f) e^{2\pi i f t}\, df$$

<u>All</u> $\mathcal{L}_2$ functions have Fourier transforms in this sense.

$\mathcal{L}_2$ waveforms do not include some favorite waveforms of most EE's.

Constants, sine waves, impulses are all infinite energy "functions."

Constants and sine waves result from refusing to explicitly model when "very long-lasting" functions terminate. Impulses result from refusing to model how long very short pulses last. Both models ignore energy.

These are useful models for many problems.

As communication waveforms, infinite energy waveforms make MSE quantization results meaningless. Also they make most channel results meaningless.

The DTFT (Discrete-time Fourier transform) is the t↔f dual of the Fourier series.

Theorem (DTFT) Assume $\{\hat{u}(f) : [-W, W] \to \mathbb{C}\}$ **is** $\mathcal{L}_2$ **(and thus also** $\mathcal{L}_1$**). Then**

$$u_k = \frac{1}{2W} \int_{-W}^{W} \hat{u}(f) e^{2\pi i k f/(2W)} \, df$$

is a finite complex number for each $k \in \mathbb{Z}$**. Also**

$$\lim_{k_0 \to \infty} \int_{-W}^{W} \left| \hat{u}(f) - \sum_{k=-k_0}^{k_0} u_k \, e^{-2\pi i k f/(2W)} \right|^2 df = 0,$$

$$\hat{u}(f) = \text{l.i.m.} \sum_k u_k e^{-2\pi i f t/(2W)} \text{rect}\left(\frac{f}{2W}\right)$$

The DTFT is the same as the Fourier series, interchanging t and f, and replacing T by $2W$ and $e^{2\pi\cdots}$ by $e^{-2\pi\cdots}$.

Also, as with the Fourier series, for any set of $\{u_k; k \in \mathbb{Z}\}$ such that $\sum_k |u_k^2| < \infty$, there is a frequency function satisfying

$$\hat{u}(f) = \text{l.i.m.} \sum_k u_k e^{-2\pi i f t/(2W)} \text{rect}\left(\frac{f}{2W}\right)$$

Also, the energy equation holds,

$$\int_{-W}^{W} |\hat{u}(f)|^2 \, df = 2W \sum_{k=-\infty}^{\infty} |u_k|^2.$$

For any $\mathcal{L}_2$ function $\{\hat{u}(f) : [-W, W] \to \mathbb{C}\}$

$$\hat{u}(f) = \text{l.i.m.} \sum_k u_k e^{-2\pi i k f/(2W)} \text{rect}\left(\frac{f}{2W}\right)$$

We also write this as

$$\hat{u}(f) = \text{l.i.m.} \sum_k u_k \hat{\phi}_k(f), \quad \textbf{where}$$

$$\hat{\phi}_k(f) = e^{-2\pi i k f/(2W)} \text{rect}(\frac{f}{2W})$$

An $\mathcal{L}_2$ waveform $u(t)$ is said to be a baseband waveform of bandwidth W Hz if $\hat{u}(f) = 0$ for $|f| > W$. The transform $\hat{u}(f)$ then has both an inverse Fourier transform and a DTFT.

Note that $\hat{u}(f)$ for an $\mathcal{L}_2$ baseband waveform is both $\mathcal{L}_1$ and $\mathcal{L}_2$. Thus, at every t

$$u(t) = \int_{-W}^{W} \hat{u}(f) e^{2\pi i f t} \, df \tag{3}$$

and $u(t)$ is continuous.

Since $\hat{u}(f) = \sum u_k \hat{\phi}_k(f)$, we have $u(t) = \sum u_k \phi_k(t)$ where

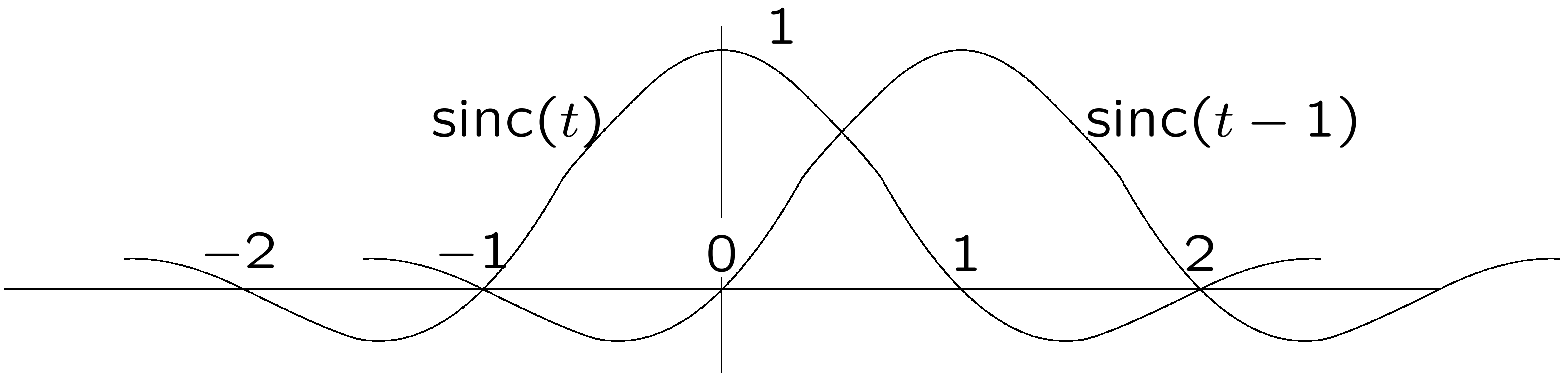
$$\hat{\phi}_k(f) = e^{-2\pi i k f/(2W)} \text{rect}(\frac{r}{2W})$$

$$\phi_k(t) = 2W \text{sinc}(2Wt - k)$$

Then

$$u(t) = \sum_k u_k \phi_k(t) = \sum_k 2W u_k \text{sinc}(2Wt - k)$$

$$u(t) = \sum_k 2W u_k \text{sinc}(2Wt - k)$$



Note that $2Wu_k = u(k/(2W))$**. Thus**

$$u(t) = \sum_{k=-\infty}^{\infty} u(\frac{k}{2W}) \ \text{sinc}\,(2Wt - k)\,.$$

Sampling Theorem: Let $\{\hat{u}(f) : [-W\,W] \to \mathbb{C}\}$ **be** $\mathcal{L}_2$ **(and thus also** $\mathcal{L}_1$**). For** $u(t)$ **in (3), let** $T = 1/(2W)$**. Then** $u(t)$ **is continuous,** $\mathcal{L}_2$**, and bounded by** $u(t) \leq \int_{-W}^{W} |\hat{u}(f)|\, df$**. Also, for all** $t \in \mathbb{R}$**,**

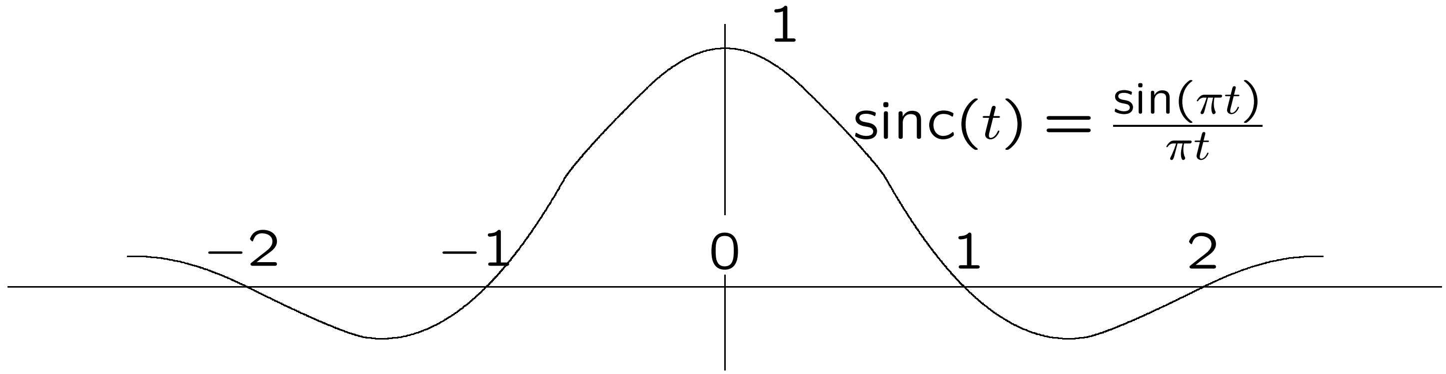
$$u(t) = \sum_{k=-\infty}^{\infty} u(kT)\ \mathrm{sinc}\left(\frac{t - kT}{T}\right).$$

Note that we started with $\{\hat{u}(f) : [-W, W] \to \mathbb{C}\}$.

There are other bandlimited functions, limited to $[-W, W]$, which are not continuous. That is, there are functions that are $\mathcal{L}_2$ equivalent to the $u(t)$ above. They have the same FT, but they are different on any arbitrary set of points of measure 0.

The sampliing theorem does not hold for these functions.

Baseband limited to W, from now on, means the continuous function whose Fourier transform is limited to $[-W, W]$.



To see why the sampling theorem is true (and also to understand the DTFT, note that

$$\hat{u}(f) = \sum_k u_k \hat{\phi}_k(f) \quad \leftrightarrow \quad u(t) = \sum_k u_k \phi_k(t).$$

$$\hat{\phi}_k(f) = e^{-2\pi i k f T}\text{rect}(fT) \ \leftrightarrow \ \phi_k(t) = \frac{1}{T}\text{sinc}(\frac{t-kT}{T})$$

Finally, $u(kT) = \frac{1}{T}u_k$**. Thus the DTFT coefficients are just scaled samples of** $u(t)$**.**