Measure and complements

We listed the rational numbers in $[-T/2, T/2]$ **as** $a_1, a_2, \ldots$

$$\mu\{\bigcup_{i=1}^{k} a_i\} = \sum_{i=1}^{k} \mu([a_i, a_i]) = 0$$

The complement of $\bigcup_{i=1}^{k} a_i$ **is** $\bigcap_{i=1}^{k} \overline{a}_i$ **where** $\overline{a}_i$ **is all** $t \in [-T/2.T/2]$ **except** a_i**.**

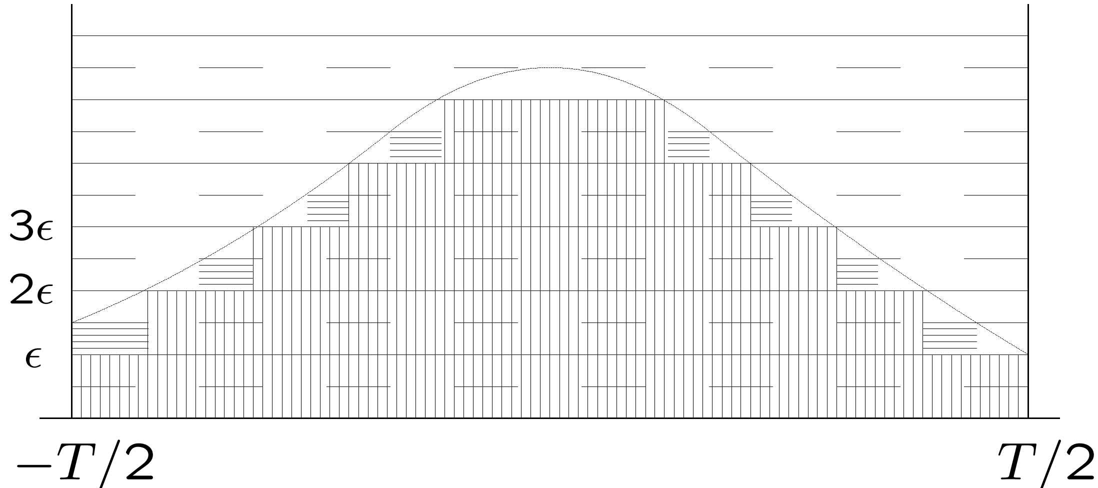
Thus $\bigcap_{i=1}^{k} \overline{a}_i$ **is a union of** $k{+}1$ **intervals, filling** $[-T/2, T/2]$ **except** $a_1, \ldots, a_k$**.**

In the limit, this is the union of an uncountable set of irrational numbers; the measure is T**.**

MEASURABLE FUNCTIONS

A function $\{u(t) : \mathbb{R} \to \mathbb{R}\}$ is measurable if $\{t : u(t) < b\}$ is measurable for each $b \in \mathbb{R}$.

The Lebesgue integral exists if the function is measurable and if the limit in the figure exists.

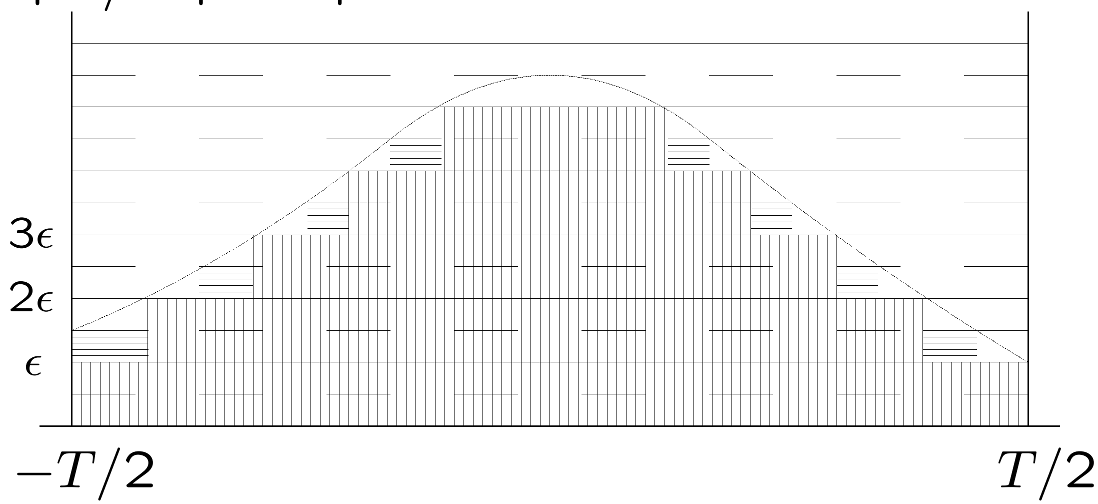


Horizontal crosshatching is what is added when $\varepsilon \to \varepsilon/2$. For $u(t) \geq 0$, the integral must exist (with perhaps an infinite value).

For $u(t) \geq 0$, the Lebesgue approximation might be infinite for all ε. Example: $u(t) = |1/t|$.

If approximation finite for any ε, then changing ε to $\varepsilon/2$ adds at most $\varepsilon/2$ to approximation.

Continued halving of interval adds at most $\varepsilon/2 + \varepsilon/4 + \cdots + \rightarrow \varepsilon$.



If any approximation is finite, integral is finite.

For a positive and negative function $u(t)$ define a positive and negative part:

$$u^+(t) = \begin{cases} u(t) & \textbf{for } t : u(t) \geq 0 \\ 0 & \textbf{for } t : u(t) < 0 \end{cases}$$

$$u^-(t) = \begin{cases} 0 & \textbf{for } t : u(t) \geq 0 \\ -u(t) & \textbf{for } t : u(t) < 0. \end{cases}$$

$$u(t) = u^+(t) - u^-(t).$$

If $u(t)$ is measurable, then $u^+(t)$ and $u^-(t)$ are also and can be integrated as before.

$$\int u(t) = \int u^+(t) - \int u^-(t)\, dt.$$

except if both $\int u^+(t)\, dt$ and $\int u^-(t)\, dt$ are infinite, then the integral is undefined.

For $\{u(t) : [-T/2, T/2] \to \mathbb{R}\}$**, the functions** $|u(t)|$ **and** $|u(t)|^2$ **are non-negative.**

They are measurable if $u(t)$ **is.**

$|u(t)| = u^+(t) + u^-(t)$ **thus** $\int |u(t)|\,dt = \int u^+(t)\,dt + \int u^-(t)\,dt$

Def: $u(t)$ **is** $\mathcal{L}_1$ **if measurable and** $\int |u(t)|\,dt < \infty$**.**

Def: $u(t)$ **is** $\mathcal{L}_2$ **if measurable and** $\int |u(t)|^2\,dt < \infty$**.**

A complex function $\{u(t) : [-T/2, T/2] \to \mathbb{C}\}$ **is measurable if both** $\Re[u(t)]$ **and** $\Im[u(t)$ **are measurable.**

Def: u(t) is $\mathcal{L}_1$ **if** $\int |u(t)|\, dt < \infty$.

Since $|u(t)| \leq |\Re(u(t)| + |\Im(u(t)|$**, it follows that** $u(t)$ **is** $\mathcal{L}_1$ **if and only if** $\Re[u(t)]$ **and** $\Im[u(t)]$ **are** $\mathcal{L}_1$**.**

Def: $u(t)$ **is** $\mathcal{L}_2$ **if** $\int |u(t)|^2\, dt < \infty$**. This happens if and only if** $\Re[u(t)]$ **and** $\Im[u(t)]$ **are** $\mathcal{L}_2$**.**

If $|u(t)| \geq 1$ **for given** t**, then** $|u(t)| \leq |u(t)|^2$**.**

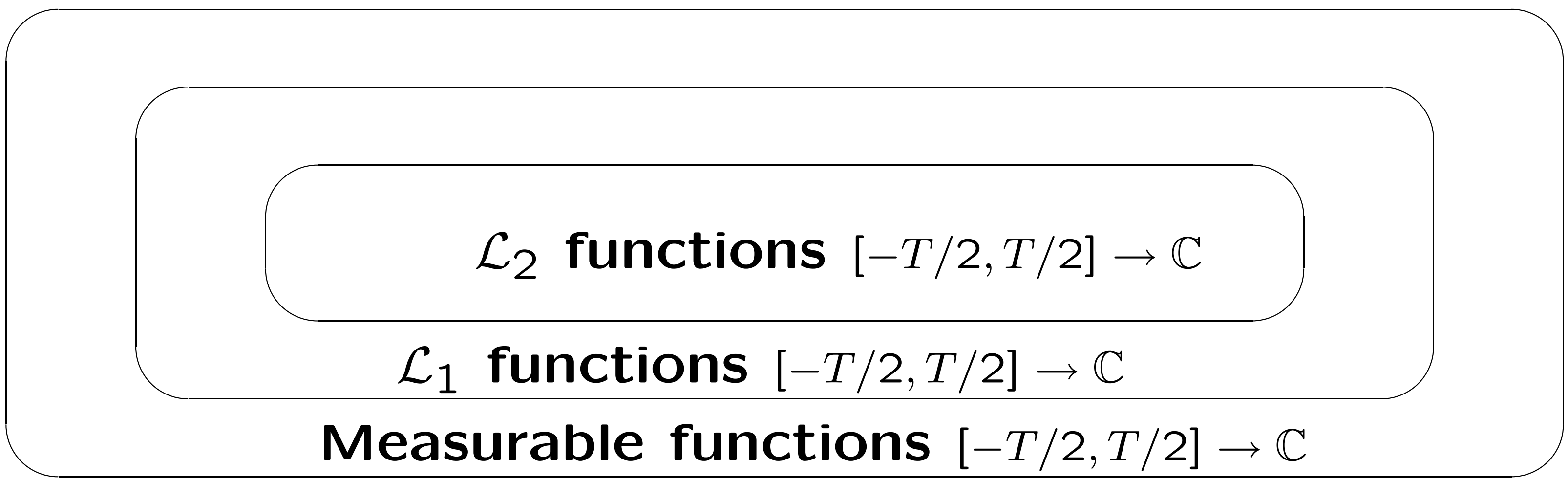
Otherwise $|u(t)| \leq 1$**. For all** t**,**

$$|u(t)| \leq |u(t)|^2 + 1.$$

For $\{u(t) : [-T/2, T/2 \to \mathbb{C}]$**,**

$$\begin{aligned} \int_{-T/2}^{T/2} |u(t)|\, dt &\leq \int_{-T/2}^{T/2} [|u(t)|^2 + 1]\, dt \\ &= T + \int_{-T/2}^{T/2} |u(t)|^2\, dt \end{aligned}$$

Thus $\mathcal{L}_2$ **finite duration functions are also** $\mathcal{L}_1$**.**


$\mathcal{L}_2$ **functions** $[-T/2, T/2] \to \mathbb{C}$
$\mathcal{L}_1$ **functions** $[-T/2, T/2] \to \mathbb{C}$
Measurable functions $[-T/2, T/2] \to \mathbb{C}$

Back to Fourier series:

Note that $|u(t)| = |u(t)e^{2\pi ift}|$

Thus, if $\{u(t) : [-T/2, T/2] \to \mathbb{C}\}$ **is** $\mathcal{L}_1$**, then**

$$\int |u(t)e^{2\pi ift}|\, dt < \infty.$$

$$|\int u(t)e^{2\pi ift}\, dt| \leq \int |u(t)|\, dt < \infty.$$

If $u(t)$ **is** $\mathcal{L}_2$ **and time-limited, it is** $\mathcal{L}_1$ **and same conclusion follows.**

Theorem: Let $\{u(t) : [-T/2, T/2] \to \mathbb{C}\}$ **be an** $\mathcal{L}_2$ **function. Then for each** $k \in \mathbb{Z}$**, the Lebesgue integral**

$$\hat{u}_k = \frac{1}{T} \int_{-T/2}^{T/2} u(t)\, e^{-2\pi ikt/T}\, dt$$

exists and satisfies $|\hat{u}_k| \le \frac{1}{T} \int |u(t)|\, dt < \infty$**. Furthermore,**

$$\lim_{k_0 \to \infty} \int_{-T/2}^{T/2} \left| u(t) - \sum_{k=-k_0}^{k_0} \hat{u}_k\, e^{2\pi ikt/T} \right|^2 dt = 0,$$

where the limit is monotonic in k_0**.**

The most important part of the theorem is that

$$u(t) \approx \sum_{k=-k_0}^{k_0} \hat{u}_k e^{2\pi i k t/T}$$

where the energy difference between the terms goes to 0 as $k_0 \to \infty$, i.e.,

$$\lim_{k_0 \to \infty} \int_{-T/2}^{T/2} \left| u(t) - \sum_{k=-k_0}^{k_0} \hat{u}_k \, e^{2\pi i k t/T} \right|^2 dt = 0,$$

We abbreviate this convergence by

$$u(t) = \text{l.i.m.} \sum_k \hat{u}_k \, e^{2\pi i k t/T} \, \text{rect}(\frac{t}{T}).$$

$$u(t) = \text{l.i.m.} \sum_k \hat{u}_k \, e^{2\pi ikt/T} \, \text{rect}(\frac{t}{T}).$$

This does not mean that the sum on the right converges to $u(t)$ at each t and does not mean that the sum converges to anything.

There is an important theorem by Carleson that says that for $\mathcal{L}_2$ functions, the sum converges a.e. That is, it converges to $u(t)$ except on a set of t of measure 0.

This means that it converges for all integration purposes.

It is often important to go from sequence to function. The relevant result about Fourier series then is

Theorem: If a sequence of complex numbers $\{\hat{u}_k; k \in \mathbb{Z}\}$ **satisfies** $\sum_k |\hat{u}_k|^2$**, then an** $\mathcal{L}_2$ **function** $\{u(t) : [-T/2, T/2] \to \mathbb{C}\}$ **exists satisfying**

$$u(t) = \text{l.i.m.} \sum_k \hat{u}_k \, e^{2\pi i k t/T} \, \text{rect}(\frac{t}{T}).$$

Aside from all the mathematical hoopla (which is important), there is a very simple reason why so many things are simple with Fourier series. The expansion functions,

$$\theta_k(t) = e^{2\pi ikt/T}\,\mathrm{rect}(t/T)$$

are orthogonal. That is

$$\int \theta_k(t)\theta_j^*(t)\,dt = T\delta_{k,j}$$

This is the feature that let us solve for $\hat{u}_k(t)$ **from the Fourier series** $u(t) = \sum_k \hat{u}_k\theta_k(t)$**.**

Functions not limited in time

We can segment an arbitrary $\mathcal{L}_2$ function into segments of width T. The mth segment is $u_m(t) = u(t)\text{rect}(t/T - m)$. We then have

$$u(t) = \text{l.i.m.}_{m_0\to\infty} \sum_{m=-m_0}^{m_0} u_m(t)$$

This works because $u(t)$ is $\mathcal{L}_2$. The energy in $u_m(t)$ must go to 0 as $m \to \infty$.

By shifting $u_m(t)$, we get the Fourier series:

$$u_m(t) \;=\; \text{l.i.m.} \sum_k \hat{u}_{k,m}\, e^{2\pi ikt/T}\, \text{rect}(\frac{t}{T} - m), \qquad \textbf{where}$$

$$\hat{u}_{k,m} \;=\; \frac{1}{T}\int_{-\infty}^{\infty} u(t)e^{-2\pi ikt/T}\text{rect}(\frac{t}{T} - m)\, dt, \;\; -\infty < k < \infty.$$

This breaks $u(t)$ into a double sum expansion of orthogonal functions, first over segments, then over frequencies.

$$u(t) = \text{l.i.m.} \sum_{m,k} \hat{u}_{k,m}\, e^{2\pi ikt/T}\, \text{rect}(\frac{t}{T} - m)$$

This is the first of a number of orthogonal expansions of arbitrary $\mathcal{L}_2$ functions.

We call this the T-spaced truncated sinusoid expansion.

$$u(t) = \text{l.i.m.} \sum_{m,k} \hat{u}_{k,m}\, e^{2\pi i k t/T}\, \text{rect}(\frac{t}{T} - m)$$

This is the conceptual basis for algorithms such as voice compression that segment the waveform and then process each segment.

It matches our intuition about frequency well; that is, in music, notes (frequencies) keep changing.

The awkward thing is that the segmentation parameter T is arbitrary and not fundamental.

Fourier transform: $u(t) : \mathbb{R} \to \mathbb{C}$ **to** $\hat{u}(f) : \mathbb{R} \to \mathbb{C}$

$$\hat{u}(f) = \int_{-\infty}^{\infty} u(t) e^{-2\pi i f t} \, dt.$$

$$u(t) = \int_{-\infty}^{\infty} \hat{u}(f) e^{2\pi i f t} \, df.$$

For "well-behaved functions," first integral exists for all f**, second exists for all** t **and results in original** $u(t)$**.**

What does well-behaved mean? It means that the above is true.

$au(t) + bv(t)$	$\leftrightarrow$	$a\hat{u}(f) + b\hat{v}(f).$	**Linearity**
$u^*(-t)$	$\leftrightarrow$	$\hat{u}^*(f).$	**Conjugate**
$\hat{u}(t)$	$\leftrightarrow$	$u(-f).$	**Duality**
$u(t-\tau)$	$\leftrightarrow$	$e^{-2\pi i f \tau}\hat{u}(f)$	**Time shift**
$u(t)e^{2\pi i f_0 t}$	$\leftrightarrow$	$\hat{u}(f - f_0)$	**Frequency shift**
$u(t/T)$	$\leftrightarrow$	$T\,\hat{u}(fT).$	**Scaling**
$du(t)/dt$	$\leftrightarrow$	$i2\pi f\hat{u}(f).$	**Differentiation**
$\int_{-\infty}^{\infty} u(\tau)v(t-\tau)\,d\tau$	$\leftrightarrow$	$\hat{u}(f)\hat{v}(f).$	**Convolution**
$\int_{-\infty}^{\infty} u(\tau)v^*(\tau-t)\,d\tau$	$\leftrightarrow$	$\hat{u}(f)\hat{v}^*(f).$	**Correlation**

Two useful special cases of any Fourier transform pair are:

$$u(0) = \int_{-\infty}^{\infty} \hat{u}(f)\, df;$$

$$\hat{u}(0) = \int_{-\infty}^{\infty} u(t)\, dt.$$

Parseval's theorem:

$$\int_{-\infty}^{\infty} u(t)v^*(t)\, dt = \int_{-\infty}^{\infty} \hat{u}(f)\hat{v}^*(f)\, df.$$

Replacing $v(t)$ **by** $u(t)$ **yields the energy equation,**

$$\int_{-\infty}^{\infty} |u(t)|^2\, dt = \int_{-\infty}^{\infty} |\hat{u}(f)|^2\, df.$$

6.450 Principles of Digital Communication I
Fall 2009