Spring 2016
6.441 - Information Theory
Final
Due: Tue, May 17, 2016 by midnight (email or office mailbox)
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1 Rules

1. Collaboration strictly prohibited.
2. Write rigorously, prove all claims.
3. You can use notes and textbooks.
4. All exercises are 10 points.

2 Exercises

1 (Compression under noisy observation) Let P_S be a discrete memoryless source on alphabet $\mathcal{A}$. Consider a reconstruction alphabet $\hat{\mathcal{A}}$ and a distortion $d(a, \hat{a})$. Suppose that compressor only observes S through a noisy channel $P_{Y|S}$ (i.e. encoder is a map $f : \mathcal{Y}^n \to [M]$). Show that the rate-distortion function is given by

$$R(D) = \min I(Y; \hat{S}) ,$$

where minimization is over all $P_{\hat{S}|Y}$ satisfying

$$\mathbb{E}\,[d(S, \hat{S})] \le D ,$$

and the probability space is $S \to Y \to \hat{S}$. (Hint: think of $\mathbb{E}\,[d(S, \hat{a})|Y = y]$.)

2 Let $S_j \in \{\pm 1\}$ be a stationary two-state Markov process with

$$P_{S_j|S_{j-1}}(s|s') = \begin{cases} \tau, s \neq s' \\ 1 - \tau, s = s' . \end{cases}$$

Let $E_j \stackrel{iid}{\sim} \mathrm{Ber}(\delta)$, with $E_j \in \{0, 1\}$ and let Y_j be the observation of S_j through the binary erasure channel with erasure probability δ, i.e.

$$Y_j = S_j E_j .$$

Find entropy rate of Y_j (you can give answer in the form of a convergent series). Evaluate at $\tau = 0.11$, $\delta = 1/2$ and compare with $H(Y_1)$.

3 (Input-output cost) Let $P_{Y|X} : \mathcal{X} \to \mathcal{Y}$ be a DMC and consider a cost function $\mathsf{c} : \mathcal{X} \times \mathcal{Y} \to \mathbb{R}$ (note that $\mathsf{c}(x, y) \le L < \infty$ for some L). Consider a problem of channel coding, where the error-event is defined as

$$\{\text{error}\} \stackrel{\triangle}{=} \{\hat{W} \neq W\} \cup \left\{ \sum_{k=1}^{n} \mathsf{c}(X_k, Y_k) > nP \right\} ,$$

where P is a fixed parameter. Define operational capacity $C(P)$ and show it is given by

$$C_i(P) = \max_{P_X : \mathbb{E}[\mathsf{c}(X,Y)] \le P} I(X;Y)$$

(Hint: do a converse directly, and for achievability reduce to an appropriately chosen cost-function $\mathsf{c}'(x)$).

4 Consider a DMC with two outputs $P_{Y,U|X}$. Suppose that receiver observes only Y, while U is (causally) fed back to the transmitter. We know that when $Y = U$ the capacity is not increased.

1. Show that capacity is not increased in general (even when $Y \neq U$).
2. Suppose now that there is a cost function c and $\mathsf{c}(x_0) = 0$. Show that capacity per unit cost (with U being fed back) is still given by

$$C_V = \max_{x \neq x_0} \frac{D(P_{Y|X=x} \| P_{Y|X=x_0})}{\mathsf{c}(x)}$$

5 (Capacity of sneezing) A sick student is sneezing periodically every minute, with each sneeze happening i.i.d. with probability p. He decides to send k bits to a friend by modulating the sneezes. For that, every time he realizes he is about to sneeze he chooses to suppress a sneeze or not. A friend listens for n minutes and then tries to decode k bits.

1. Find capacity in bits per minute. (Hint: Think how to define the channel so that channel input at time t were not dependent on the arrival of the sneeze at time t. To rule out strategies that depend on arrivals of past sneezes, you may invoke Exercise 4.)
2. Suppose sender can suppress at most E sneezes and listener can wait indefinitely ($n = \infty$). Show that sender can transmit $C_{puc}E + o(E)$ bits reliably as $E \to \infty$ and find C_{puc}. Curiously, $C_{puc} \geq 1.44$ bits/sneeze regardless of p. (Hint: This is similar to Exercise 3.)