Spring 2016
6.441 - Information Theory
Homework 6
Due: Thur, Mar 17, 2016 (in class)
`Prof. Y. Polyanskiy`

1 Reading (optional)

1. Read [1, Chapters 11,12]

2 Exercises

NOTE: Each exercise is 10 points. Only 3 exercises per assignment will be graded. If you submit more than 3 solved exercises please indicate which ones you want to be graded.

1 Consider a binary hypothesis testing problem:

$$\mathrm{H}_0 \quad : \quad X_1, \ldots, X_n - \text{ i.i.d. with } X_j \sim P_0 \tag{1}$$
$$\mathrm{H}_1 \quad : \quad X_1, \ldots, X_n - \text{ i.i.d. with } X_j \sim P_1 \tag{2}$$

For $n = 1$ the hypothesis testing region

$$\mathcal{R}(P_0, P_1) \triangleq \bigcup_{P_{Z|X_1}} (\mathbb{P}_0[Z = 0], \mathbb{P}_1[Z = 0])$$

is shown on Fig. 1. Thus, if one wants reliability $\alpha \triangleq \pi_{0|0} = 0.99$ then the smallest probability of error $\beta \triangleq \pi_{0|1}$ is ≈ 0.993. By taking more observations (i.e., $n \gg 1$) the quality of the test can be dramatically improved. Estimate minimal n required to achieve

$$\pi_{0|0} \quad \geq \quad 0.99 \tag{3}$$
$$\pi_{0|1} \quad \leq \quad 10^{-40}\,. \tag{4}$$

Describe behavior of the region $\mathcal{R}(P_0^n, P_1^n)$ as $n \to \infty$.

2 Let P be the uniform distribution on the interval $[0, 1]$. Let Q be the equal mixture of the uniform distribution on $[0, 1/2]$ and the point mass at 1.

1. Compute the the region $\mathcal{R}(P, Q)$.
2. Explicitly describe the tests that achieve the optimal boundary $\beta_\alpha(P, Q)$.

3 Recall the total variation distance

$$\mathrm{TV}(P, Q) \triangleq \sup_E \left(P[E] - Q[E]\right)\,.$$

1. Prove that
$$\mathrm{TV}(P, Q) = \sup_{0 \leq \alpha \leq 1} \{\alpha - \beta_\alpha(P, Q)\}.$$

   Explain how to read the value $\mathrm{TV}(P, Q)$ from the region $\mathcal{R}(P, Q)$.

2. (Bayesian criteria) Fix a prior $\pi = (\pi_0, \pi_1)$ such that $\pi_0 + \pi_1 = 1$ and $0 < \pi_0 < 1$. Denote the optimal Bayesian (average) error probability by

$$P_e \triangleq \inf_{P_{Z|X^n}} \pi_0 \pi_{1|0} + \pi_1 \pi_{0|1}.$$

   Prove that if $\pi = (\frac{1}{2}, \frac{1}{2})$, then

$$P_e = \frac{1}{2}(1 - \mathrm{TV}(P, Q)).$$

   Find the optimal test.

3. Find the optimal test for general prior π (not necessarily equiprobable).
4. Why is it always sufficient to focus on deteministic test in order to minimize the Bayesian error probability?

4 Function $\alpha \mapsto \beta_\alpha(P, Q)$ is monotone and thus by Lebesgue's theorem possesses a derivative

$$\beta'_\alpha \triangleq \frac{d}{d\alpha} \beta_\alpha(P, Q)\,.$$

almost everywhere on $[0, 1]$. Prove

$$D(P||Q) = -\int_0^1 \log \beta'_\alpha \, d\alpha\,.$$

5 Let P, Q be distributions such that for all $\alpha \in [0, 1]$ we have

$$\beta_\alpha(P, Q) \triangleq \min_{P_{Z|X}: P[Z=0] \geq \alpha} Q[Z = 0] = \alpha^2\,.$$

Find $\mathrm{TV}(P, Q)$, $D(P||Q)$ and $D(Q||P)$.

6 We have shown in class that for testing iid products and any fixed $\epsilon \in (0, 1)$:

$$\log \beta_{1-\epsilon}(P^n, Q^n) = -nD(P||Q) + o(n)\,, \qquad n \to \infty\,,$$

which is equivalent to Stein's lemma. Show furthermore that assuming $V(P||Q) < \infty$ we have

$$\log \beta_{1-\epsilon}(P^n, Q^n) = -nD(P||Q) + \sqrt{nV(P||Q)}Q^{-1}(\epsilon) + o(\sqrt{n})\,,$$

where $Q^{-1}(\cdot)$ is the functional inverse of $Q(x) = \int_x^\infty \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt$ and

$$V(P||Q) \triangleq \mathrm{Var}_P \left[\log \frac{dP}{dQ}\right]\,.$$

References

[1] T. Cover and J. Thomas, *Elements of Information Theory,* Second Edition, Wiley, 2006

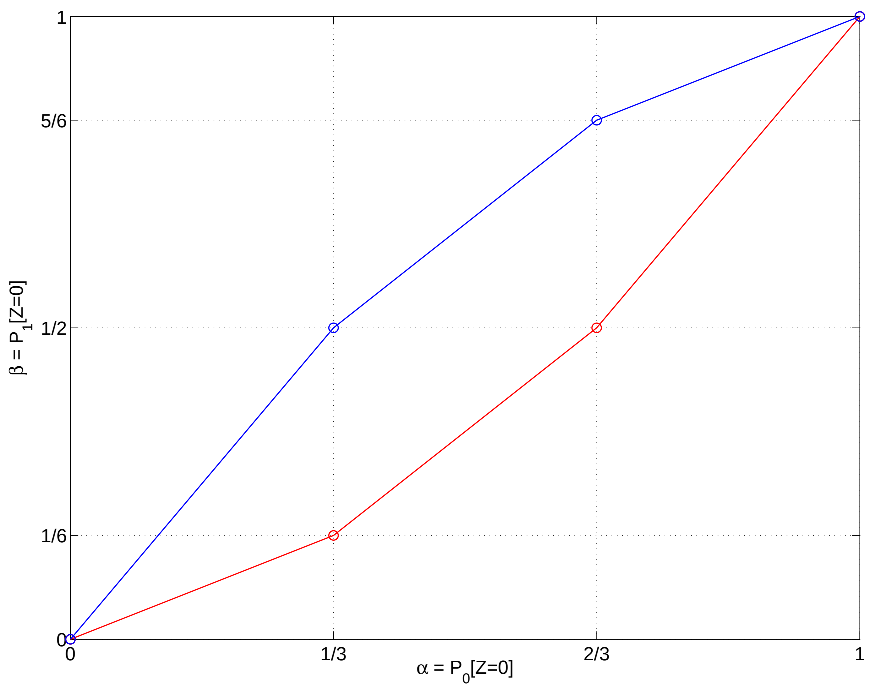

Figure 1: Upper (blue) and lower (red) boundaries of $\mathcal{R}(P_0, P_1)$ for $n = 1$

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