Spring 2016
6.441 - Information Theory
Homework 10
Due: Thur, May 5, 2016 (in class)
Prof. Y. Polyanskiy

1 Reading (optional)

1. Read [1, Chapter 10]

2 Exercises

NOTE: Each exercise is 10 points. Only 3 exercises per assignment will be graded. If you submit more than 3 solved exercises please indicate which ones you want to be graded.

1 A packet of k bits is to be delivered over an AWGN channel. To that end, a k-to-n error correcting code is used, whose probability of error is ϵ. The system employs automatic repeat request (ARQ) to resend the packet whenever an error occured.[1] Suppose that the optimal k-to-n codes achieving

$$k \approx nC - \sqrt{nV}Q^{-1}(\epsilon) + \frac{1}{2}\log n$$

are available. The goal is to optimize ϵ to get the highest average throughput: ϵ too small requires excessive redundancy, ϵ too large leads to lots of retransmissions. Compute the optimal ϵ and optimal block length n for the following four cases: SNR=0 dB or 20 dB; $k = 10^3$ or 10^4 bits.

(This gives an idea of what ϵ you should aim for in practice.)

2 Consider a binary symmetric channel with crossover probability $\delta \in (0, 1)$:

$$Y = X + Z \mod 2\,, Z \sim \text{Bern}(\delta)\,.$$

Suppose that in addition to Y the receiver also gets to observe noise Z through a binary erasure channel with erasure probability $\delta_e \in (0, 1)$. Compute:

1. Capacity C of the channel.
2. Zero-error capacity C_0 of the channel.
3. Zero-error capacity in the presence of feedback $C_{fb,0}$.

3 Consider the *polygon channel* we discussed in the lecture, where the input and output alphabet are both $\{1, \ldots, L\}$, and $P_{Y|X}(a|b) > 0$ if and only if $a = b$ or $a = (b \mod L) + 1$. *Rigorously* prove the following:

1. For all L, The zero-error capacity with feedback is $C_{fb,0} = \log \frac{L}{2}$.
2. For even L, the zero-error capacity $C_0 = \log \frac{L}{2}$.

[1] Assuming there is a way for receiver to verify whether his decoder produced the correct packet contents or not (e.g. by finding HTML tags).

4 (BEC with feedback) Consider the stationary memoryless binary erasure channel with erasure probability δ and noiseless feedback. Design a *fixed*-blocklength[2] coding scheme achieving the capacity, i.e., find a scheme that sends k bits over n channel uses with noiseless feedback, such that the rate $\frac{k}{n}$ approaches the capacity $1-\delta$ when $n \to \infty$ and the maximal probability of error vanishes. Describe the encoding and decoding operations and *rigorously* prove your result.

(Hint: Try retransmitting each bit until received.)

5 Let $\mathcal{S} = \hat{\mathcal{S}} = \{0, 1\}$ and let the source X^{10} be fair coin flips. Denote the output of the decompressor by $\hat{X}^{10}$. Show that it is possible to achieve average Hamming distortion $\frac{1}{20}$ with 512 codewords.

6 Assume the distortion function is separable. Show that the minimal number of codewords $M^*(n, D)$ required to represent memoryless source X^n with average distortion D satisfies

$$\log M^*(n_1 + n_2, D) \le \log M^*(n_1, D) + \log M^*(n_2, D)\,.$$

Conclude that

$$\lim_{n\to\infty} \frac{1}{n} \log M^*(n, D) = \inf_n \frac{1}{n} \log M^*(n, D)\,. \qquad (1)$$

(i.e. one can always achieve a better compression rate by using a longer blocklength). Neither claim holds for $\log M^*(n, \epsilon)$ in channel coding (with inf replaced by sup in (1) of course). Convince yourself you understand the reason of this different behavior.

References

[1] T. Cover and J. Thomas, *Elements of Information Theory,* Second Edition, Wiley, 2006

[2]Variable blocklength codes are not allowed.