Introductory Examples for System Identification

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Examples of Sys. Id.

Will use two systems,

$$\delta_1 : \quad y - 0.8y(t-1) = u(t-1) + e(t)$$

$$\delta_2 : \quad y - 0.8y(t-1) = u(t-1) + e(t) - 0.8e(t-1)$$

Objective

–Study the mechanics of sys. id.

–Introduce linear regressions to minimize prediction error

–Study the importance of selecting the correct model structure.

–Study the importance of selecting the input.

–Analysis: Is the theory consistent with the observations?

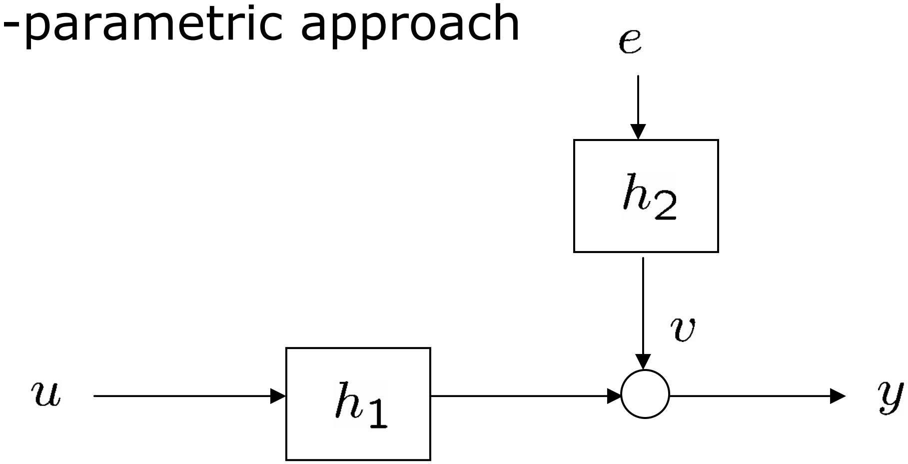
Non-parametric Methods

- Data generated with

$$e \sim N(0,1) \quad , \quad u \sim N(0,1)$$

↳ var λ^2 ↳ var σ^2

- Try a non-parametric approach



$$y = h_1 * u + h_2 * e$$

e, u are uncorrelated

- One way of recovering h_1 is to correlate the data with u and thus get rid of the noise.

- Method called: Correlation analysis

- Note: If e, u are uncorrelated , then
  v, u are uncorrelated

- The method does not estimate h_2 .

Correlation Analysis

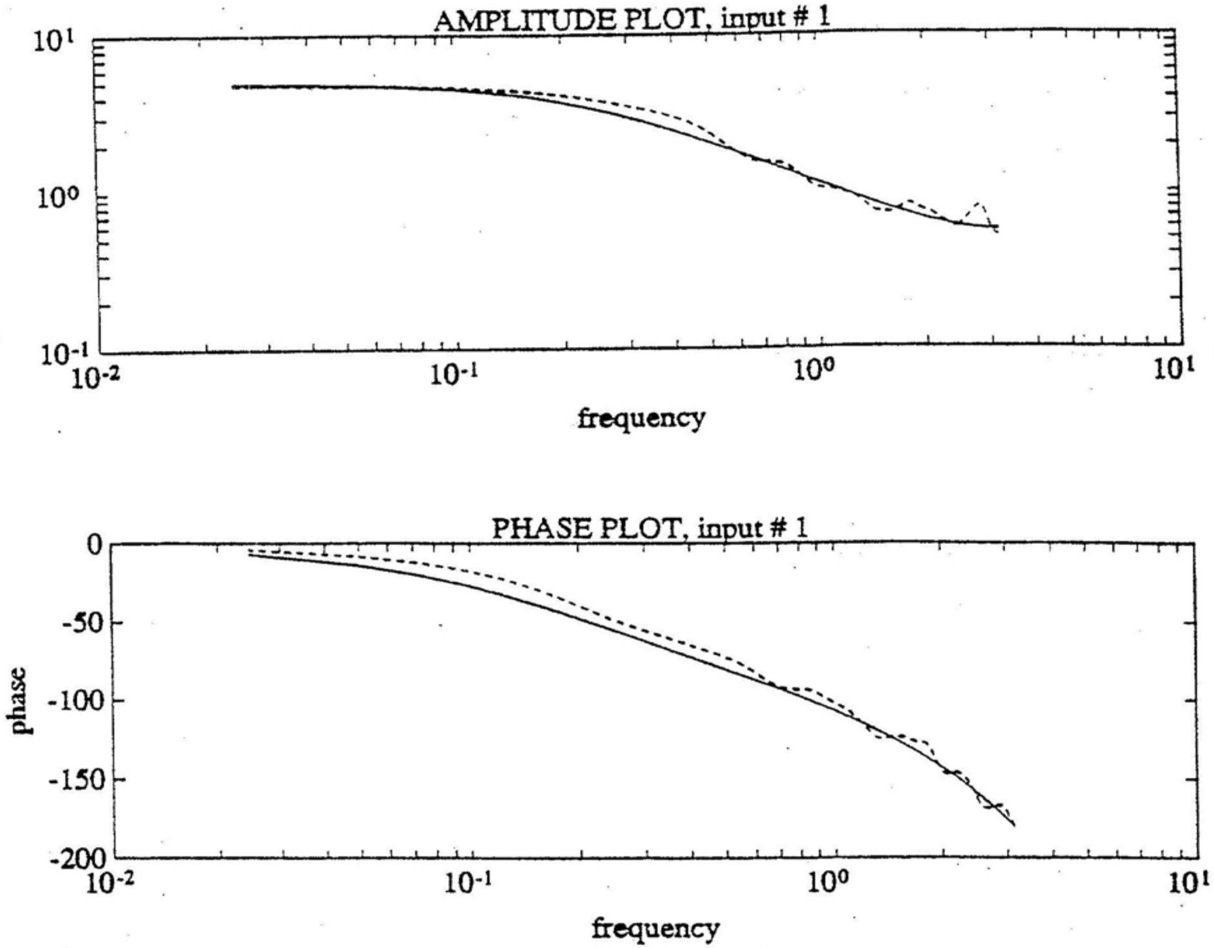
- $y(t) = \sum_{k=0}^{\infty} h(t-k)u(k) + v(k)$
- Compute $R_{yu}(\tau)$ exactly.
- $R_{yu}(\tau) = h * R_{uu}(\tau) = \sigma^2 h * \delta(t) = \sigma^2 h(\tau)$
- Approximation: Ergodicity assumptions.

$$\hat{h}(\tau) = \frac{\frac{1}{N}\sum_{t=1}^{N} y(t+\tau)u(t)}{\frac{1}{N}\sum_{t=1}^{N} u^2(t)}$$

- The Frequency response

$$\hat{H} \cong \frac{\Phi_{yu}}{\sigma^2}$$

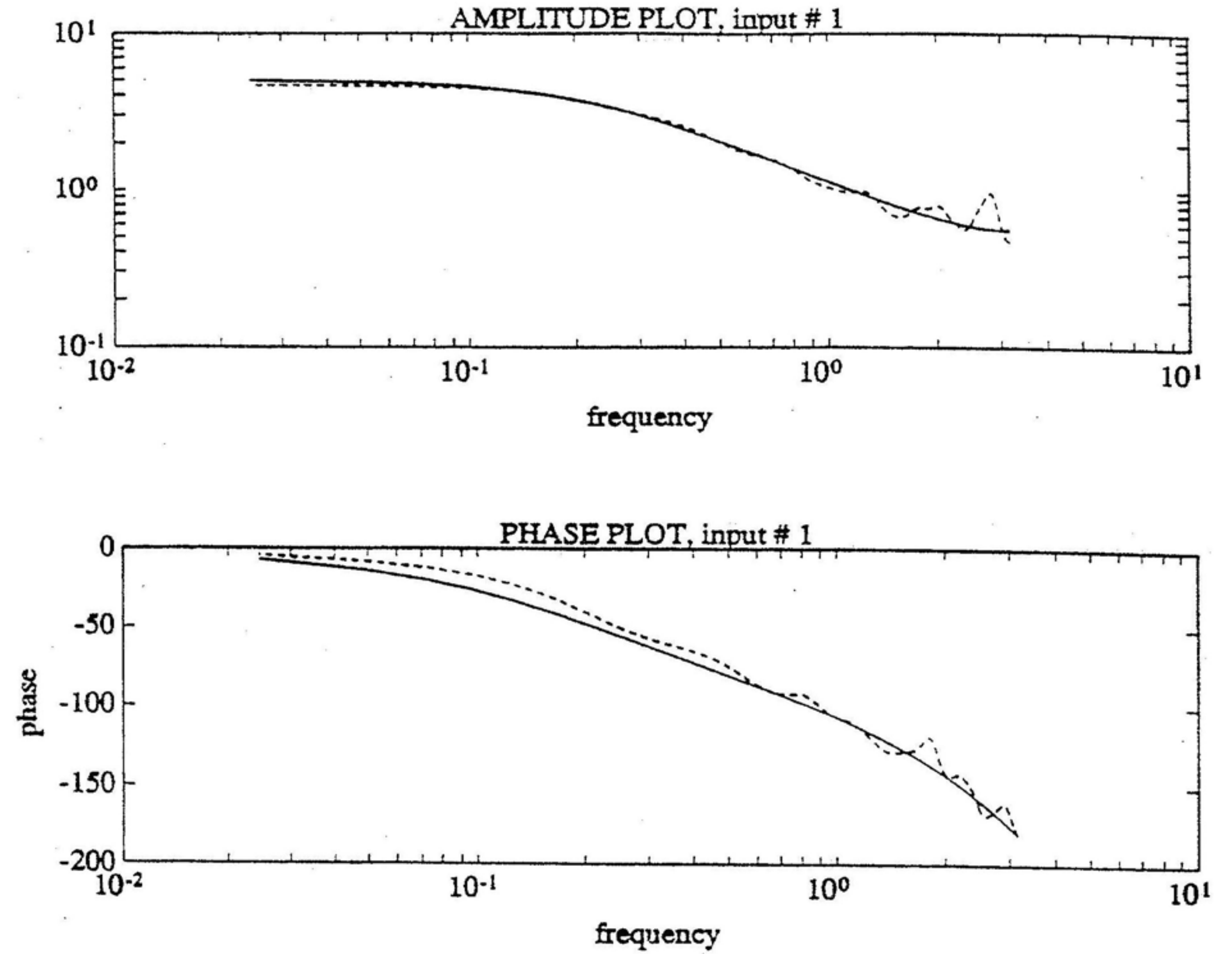
- Comparisons!



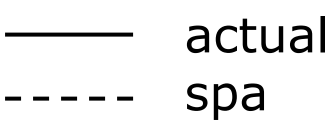
$\delta_1 : u = \text{random}$

——— actual

- - - - - spa



$\delta_2 : u =$ random



Comments

- Both systems have a “fairly” good low-freq approximation.
- For system 1, there is an error in the low frequency range.
- $\delta_1 :\quad y = \dfrac{q^{-1}}{1-0.8q^{-1}}u + \dfrac{q^{-1}}{1-0.8q^{-1}}e$

  $\delta_2 :\quad y = \dfrac{q^{-1}}{1-0.8q^{-1}}u + e$

  noise for δ_1 is amplified in the low frequency range.

A Parametric Approach

- Guess a model set

$$\underbrace{y(t) + ay(t-1) = bu(t-1)}_{\text{unknown parameter } \theta = \begin{pmatrix} a \\ b \end{pmatrix}} + \varepsilon(t)$$

$\varepsilon(t)$: disturbance describing the deviation from the real model

- Rewrite

$$\varepsilon(t) = y(t) - (-y(t-1) + u(t-1)) \begin{pmatrix} a \\ b \end{pmatrix}$$

- Pick $\widehat{\theta} = \begin{pmatrix} \widehat{a} \\ \widehat{b} \end{pmatrix}$ to minimize the mismatch between the "model" and the "physical process".

- Define $V(\theta) = \sum_{i=1}^{N} \varepsilon^2(t)$

- Estimate at time $N, \widehat{\theta}$, is given by:

$$\widehat{\theta} = \text{arg min } V(\theta)$$

Calculation of $\widehat{\theta}$

- Expand in a matrix form

$$\underbrace{\begin{bmatrix} \varepsilon(1) \\ \varepsilon(2) \\ \vdots \\ \varepsilon(N) \end{bmatrix}}_{\underline{\varepsilon}} = \underbrace{\begin{bmatrix} y(1) \\ y(2) \\ \vdots \\ y(N) \end{bmatrix}}_{\underline{y}} - \underbrace{\begin{bmatrix} -y(0) & u(0) \\ -y(1) & u(1) \\ \vdots & \vdots \\ -y(N-1) & u(N-1) \end{bmatrix}}_{\Phi} \underbrace{\begin{bmatrix} a \\ b \end{bmatrix}}_{\theta}$$

- $V(\theta) = \|\underline{\varepsilon}\|_2 = \|\underline{y} - \Phi\theta\|_2$

- $\widehat{\theta}_N = \underset{\theta}{\text{argmin}} \ \|\underline{y} - \Phi\theta\|_2$

$$= \underbrace{\left(\Phi^T\Phi\right)^{-1}\Phi^T}_{\text{pseudo-inverse}}\underline{y} \qquad \text{(see 6.241)}$$

- $\Phi^T\Phi = \begin{pmatrix} \sum y^2(t-1) & -\sum u(t-1)y(t-1) \\ -\sum u(t-1)y(t-1) & \sum u^2(t-1) \end{pmatrix}$

- $\Phi^T y = \begin{pmatrix} -\sum y(t-1)y(t) \\ \sum u(t-1)y(t) \end{pmatrix}$ all $\sum_{t=1}^{N}$

• $\widehat{\theta}_N$ satisfies

$$\begin{pmatrix} \frac{1}{N}\sum y^2(t-1) & -\frac{1}{N}\sum u(t-1)y(t-1) \\ -\frac{1}{N}\sum u(t-1)y(t-1) & \frac{1}{N}\sum u^2(t-1) \end{pmatrix} \widehat{\theta}_N = \begin{pmatrix} -\frac{1}{N}\sum y(t-1)y(t) \\ \frac{1}{N}\sum u(t-1)y(t) \end{pmatrix}$$

• If the matrix on the left is invertible, the estimate $\widehat{\theta}_N$ can be computed from the data.

• $\widehat{\theta}_N$ will be computed for both sets of data, first from δ_1, then from δ_2.

• We want to see/analyze the limiting behavior:

Whether $\widehat{\theta}_N \longrightarrow \theta_{act}$ as $N \longrightarrow \infty$

or is there a bias, and if so it is consistent with the theoretical analysis.

Simulations

- Both δ_1 & δ_2 were simulated with

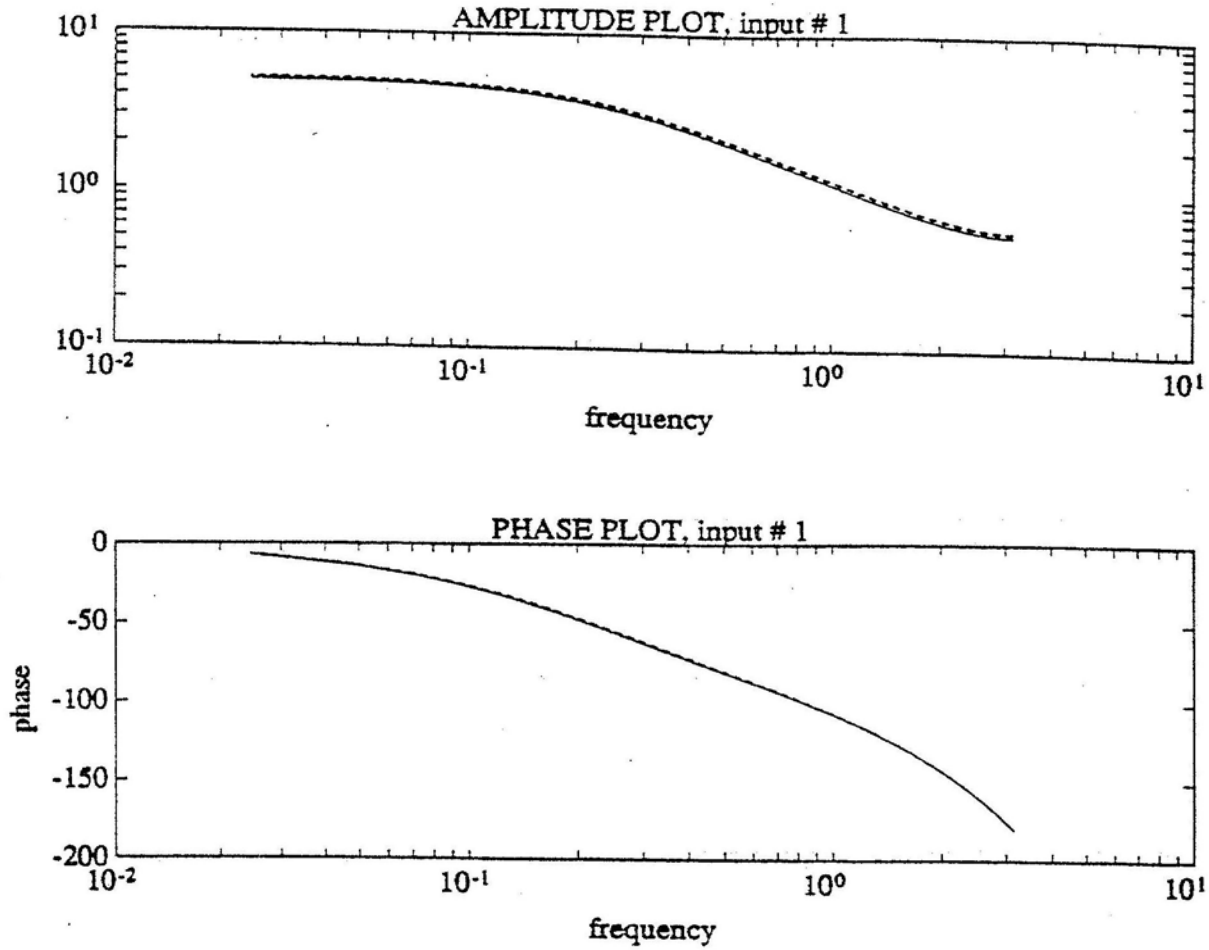
$$u = N(0,1) \quad e = N(0,1) \quad , \quad N = 100 \text{ pts}$$

- The data was used to compute $\widehat{\theta}_N$ for each system. Results are summarized below:

Parameter	True value	δ_1	δ_2
a	-0.8	-0.8149	-0.5668
b	1	+0.9885	-0.9545

- On Matlab, parameter estimation is done using "**`arx`**" command.

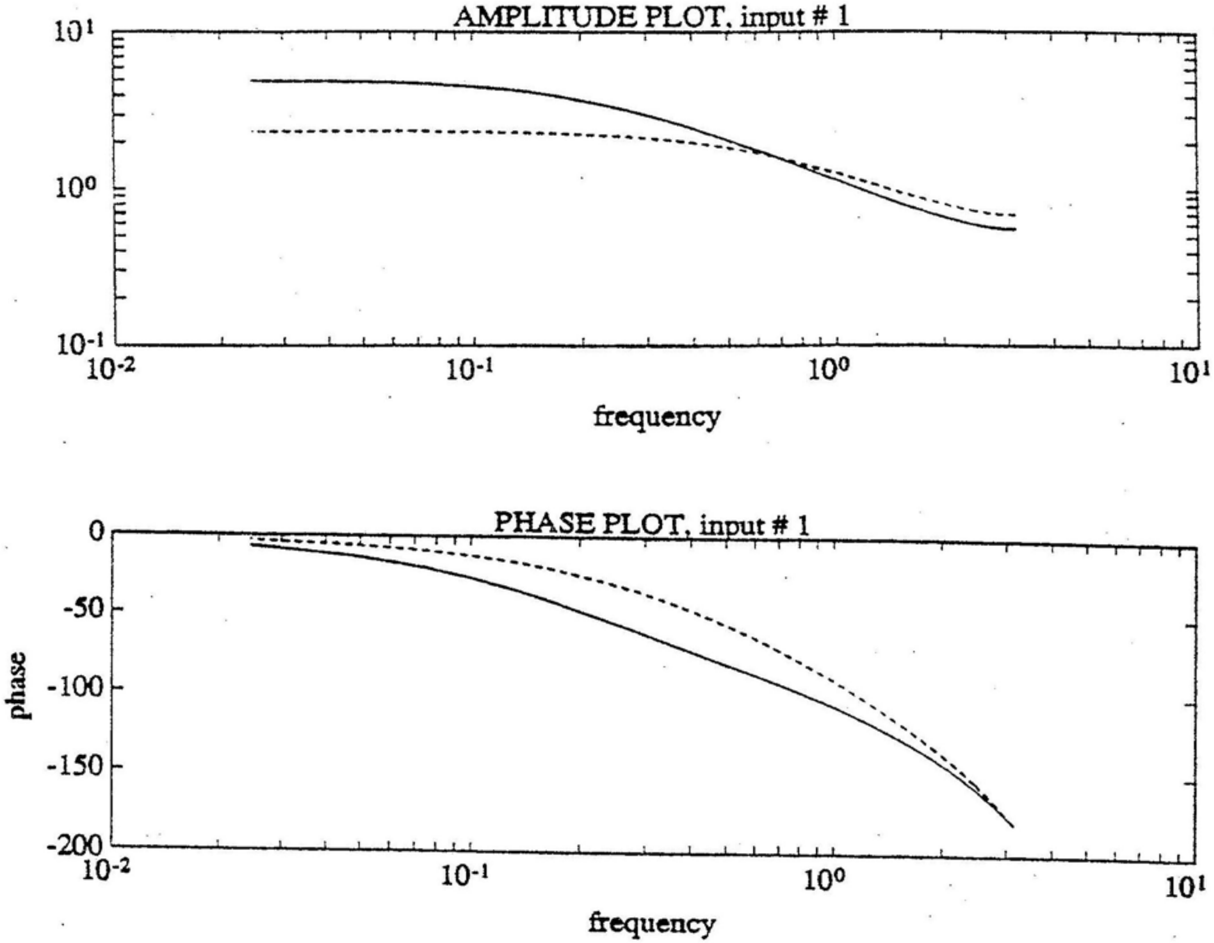
- Frequency plots follow:



$\delta_1 : u =$ random

—— actual

- - - - - arx



$\delta_2 : u =$ random

——— actual
- - - - - arx

Comments

- For System I, the model structure is identical to the model of the processes. Thus, the parameters converged to the real parameters.

- For System II, the model structure is different from the actual model. The estimated parameters had a bias.

- Is this consistent with the theoretical analysis? Are the numbers that we got meaningful or is it possible that a different simulation can yield drastically different estimated parameters?

- Theoretical analysis <u>will</u> explain the observations.

Theoretical Analysis

- Recall that $\widehat{\theta}_N$ is computed from:

$$\begin{pmatrix} \frac{1}{N}\sum y^2(t-1) & \frac{1}{N}\sum y(t-1)u(t-1) \\ \frac{1}{N}\sum y(t-1)u(t-1) & \frac{1}{N}\sum u^2(t-1) \end{pmatrix} \widehat{\theta}_N = \begin{pmatrix} \frac{1}{N}\sum y(t-1)y(t) \\ \frac{1}{N}\sum u(t-1)y(t) \end{pmatrix}$$

- Because of Ergodicity & Stationarity

$$\begin{array}{ccccc} \frac{1}{N}\sum y^2(t-1) & \simeq & E\left(y^2(t-1)\right) & = & E\left(y^2(t)\right) \\ \frac{1}{N}\sum y(t-1)u(t-1) & \simeq & E(y(t-1)u(t-1)) & = & E(y(t)u(t)) \\ \vdots & & \vdots & & \vdots \end{array}$$

- In the limit (or for N Large enough)

$$\begin{pmatrix} E\left(y^2(t-1)\right) & -E(y(t)u(t)) \\ -E(y(t)u(t)) & E\left(y^2(t-1)\right) \end{pmatrix} \widehat{\theta} = \begin{pmatrix} -Ey(t)y(t-1) \\ Ey(t)u(t-1) \end{pmatrix}$$

valid for both δ_1 & δ_2 .

The General Case

$$y(t) + ay(t-1) = bu(t-1) + e(t) + ce(t-1)$$

e, u are uncorrelated $\quad u \sim N\left(0, \sigma^2\right)$, $\quad e \sim N\left(0, \lambda^2\right)$

- Compute $Ey^2(t)$

$$Ey^2(t) = E\left(-ay(t-1) + bu(t-1) + e(t) + ce(t-1)\right)^2$$

$$= a^2 E\left(y^2(t-1)\right) + b^2 E\left(y^2(t-1)\right) + E\left(e^2(t)\right) + c^2 E\left(e^2(t-1)\right)$$

$$-2ab\underbrace{E(y(t-1)u(t-1))}_{=0} - 2a\underbrace{E(y(t-1)e(t))}_{=0} - 2acE(y(t-1)e(t-1))$$

$$+2b\underbrace{E(u(t-1)e(t))}_{=0} + 2bc\underbrace{E(u(t-1)e(t-1))}_{=0} + 2c\underbrace{E(e(t)e(t-1))}_{=0}$$

$$\left(1-a^2\right)Ey^2(t) = b^2\sigma^2+(1+c^2)\lambda^2-2acE(y(t)e(t))$$

$$= b^2\sigma^2+(1+c^2)\lambda^2-2ac\lambda^2$$

$$\Rightarrow Ey^2(t) = \frac{b^2\sigma^2+(1+c^2-2ac)\lambda^2}{1-a^2}$$

- $Ey(t)u(t) = 0$

- $Ey(t)y(t-1)$?

$$y(t)y(t-1)+ay^2(t-1) = \cancelto{0}{bu(t-1)y(t-1)}+\cancelto{0}{e(t)y(t-1)}+ce(t-1)y(t-1)$$

$$E(y(t)y(t-1))+aEy^2 = c\lambda^2$$

$$E(y(t)y(t-1)) = c\lambda^2-aEy^2$$

$$= \frac{-ab^2\sigma^2+(c-a)(1-ac)\lambda^2}{1-a^2}$$

- $Ey(t)u(t-1)$?

$$y(t)u(t-1)+ay(t-1)u(t-1) = bu^2(t-1)+e(t)u(t-1)+ce(t-1)u(t-1)$$

$$Ey(t)u(t-1) = b\sigma^2$$

- Compute the estimate $\hat{\theta}$ in the general case:

$$\hat{a} = a + \frac{-c\left(1-a^2\right)\lambda^2}{b^2\sigma^2 + (1+c^2-2ac)\lambda^2}$$

$$\hat{b} = b$$

Final Results

- For system 1, $c = 0$ and

$$\left.\begin{aligned}\widehat{a} &= a = -0.8\\ \widehat{b} &= b = 1\end{aligned}\right\}\text{ in the limit}$$

consistent with numerical results

- For system 2, $c =$ -0.8 and

$$\widehat{a} \cong -0.588$$
$$\widehat{b} = 1$$

consistent with numerical results

- Theoretical analysis is able to give very accurate explanation of the results/

- Importance of selecting the "model structure" !!

Experiment Design

- All the previous results were based on data generated $u \sim N(0,1)$ white Gaussian. (In fact, the signals are called PRBS, pseudo-random binary sequence). Such inputs are quite rich.

- How about using an input which is not very rich

  e.g. a step input

- Observe
  - Non-parametric results
  - Parametric results
  - Theoretical analysis

Step Input

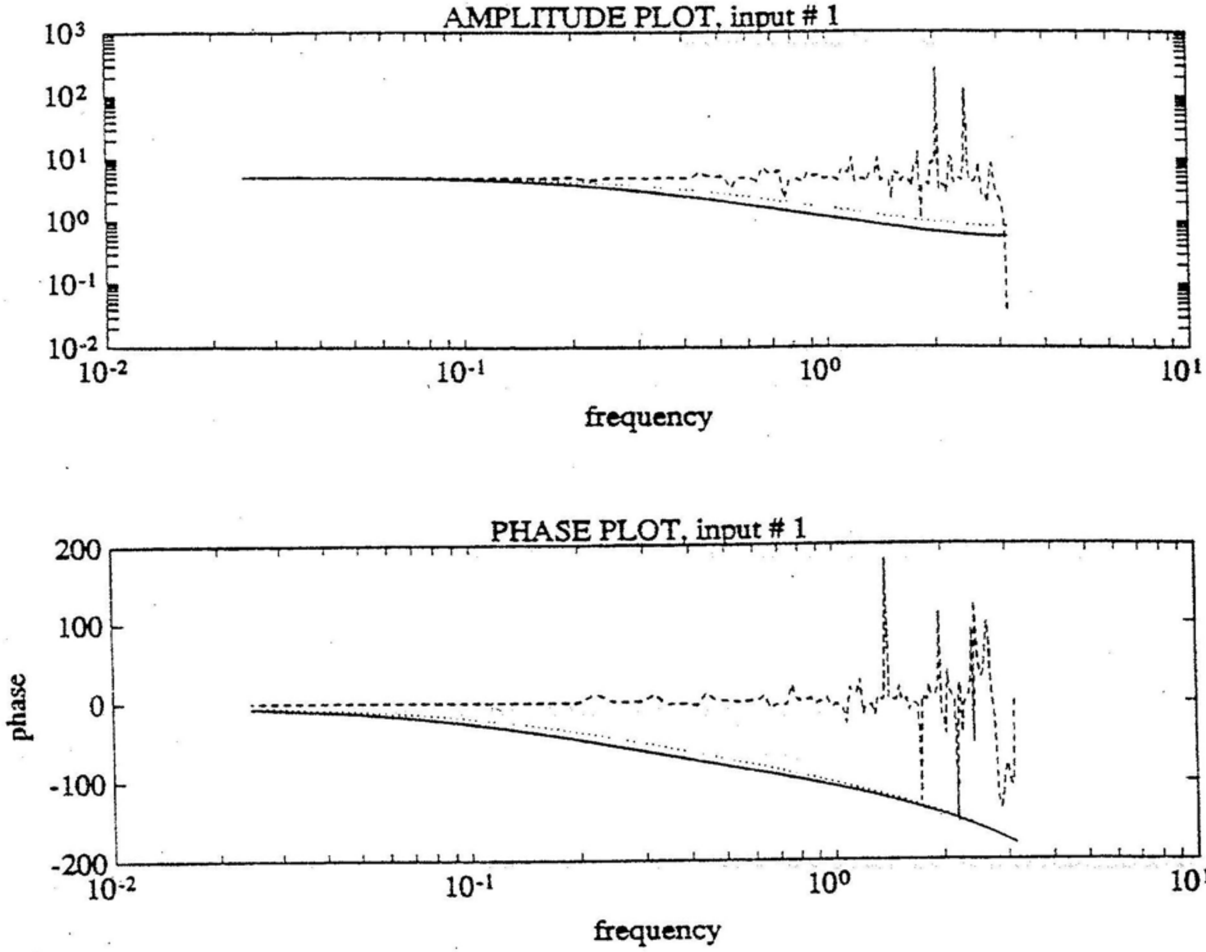
- δ_1 & δ_2 were simulated with

$$u = \text{step} \qquad e = WGN \sim N(0,1)$$

- Results from the parameter estimation are given below:

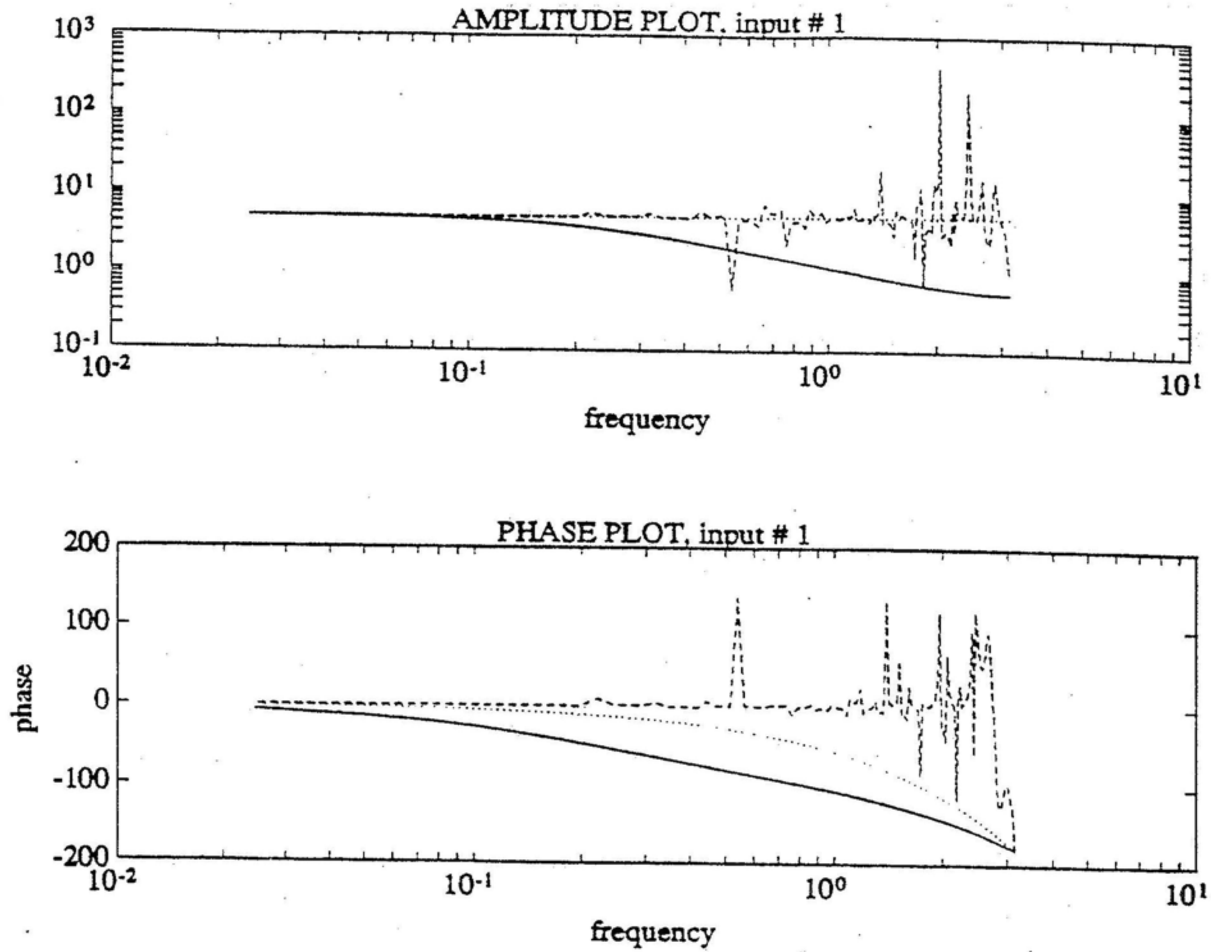
Parameter	True value	δ_1	δ_2
a	-0.8	-0.72	0.0136
b	1	1.377	5.0324

- Frequency plots follow for the actual system, the parametric estimated system, and the nonparametric estimate (based on correlation / freq methods).



δ_1, step

actual
arx
spa



δ_2, step

———	actual
·········	arx
- - - - -	spa

Comments

- Both estimates based on the frequency / correlation analysis are “bad”. A step picks up information only at “low” frequency.

Input Design is Crucial in

Non-parametric Methods !!!

- For δ_1, the parameter estimates were ”good” and seemed to be converging to the true parameters. The estimation in this case was based on the correct model structure. The “lack” of richness in the input did not seem to matter and transient response proved to be useful.

- For δ_2, the parameters were "way off", and the frequency plot is quite bad.

Lack of richness + incorrect model structure
$\Rightarrow$ bad results

- Question: Are these results dependent on the simulations? If not, can they be analyzed theoretically?

- We will show that the theoretical analysis will in fact predict this behavior.

Theoretical Analysis

- Recall

$$\begin{pmatrix} Ey^2(t) & -Ey(t)u(t) \\ -Ey(t)u(t) & Eu^2(t) \end{pmatrix} \hat{\theta} = \begin{pmatrix} -Ey(t-1)y(t) \\ Eu(t-1)y(t) \end{pmatrix}$$

- Remember $E(f(t)) \cong \frac{1}{N}\sum_{t=1}^{N} f(t)$ for deterministic signals.

$u(t) =$ step, with amplitude σ.

- In general

$$y(t) + ay(t-1) = bu(t-1) + e(t) + ce(t-1)$$

need to compute all the quantities:

$$Ey^2, Ey(t)u(t), \quad \ldots$$

- $Eu^2(t) = \sigma^2$
- $Ey^2(t) = \quad ?$

$$Ey^2(t) = a^2Ey^2(t)+b^2\sigma^2+\lambda^2+c^2\lambda^2-2abE(y(t-1)u(t-1))-2ac\lambda^2$$

$$E(y(t-1)u(t-1)) = \lim_{N\to\infty}\frac{1}{N}\sum u(t-1)y(t-1) = \sigma^2 S$$

Steady state value $S = \frac{b}{1+a}$

$$\Rightarrow Ey^2 = \sigma^2 S^2 + \frac{(1+c^2-2ac)\lambda^2}{1-a^2}$$

- $Ey(t)u(t) = S\sigma^2$
- $Ey(t)u(t-1) = S^2\sigma^2 + \frac{(c-a)(1-ac)\lambda^2}{1-a^2}$
- $Ey(t)u(t-1) = S\sigma^2$

- The estimates

$$\hat{a} = a - \frac{c\left(1-a^2\right)}{1+c^2-2ac} \qquad \hat{b} = b - bc\frac{1-a}{1+c^2-2ac}$$

- For δ_1, $\boldsymbol{c = 0}$ and

$$\left.\begin{aligned}\hat{a} &= -0.8 \\ \hat{b} &= 1\end{aligned}\right\}$$ consistent with numerical results

- For δ_2, $\boldsymbol{c = -0.8}$ and

$$\left.\begin{aligned}\hat{a} &= 0 \\ \hat{b} &= \frac{b}{1+a} = 5\end{aligned}\right\}$$ consistent with numerical results

  – Note that the s.s gain $\frac{b}{1+a}$ is correctly estimated.

- Complete agreement. Results are independent of the calculations!

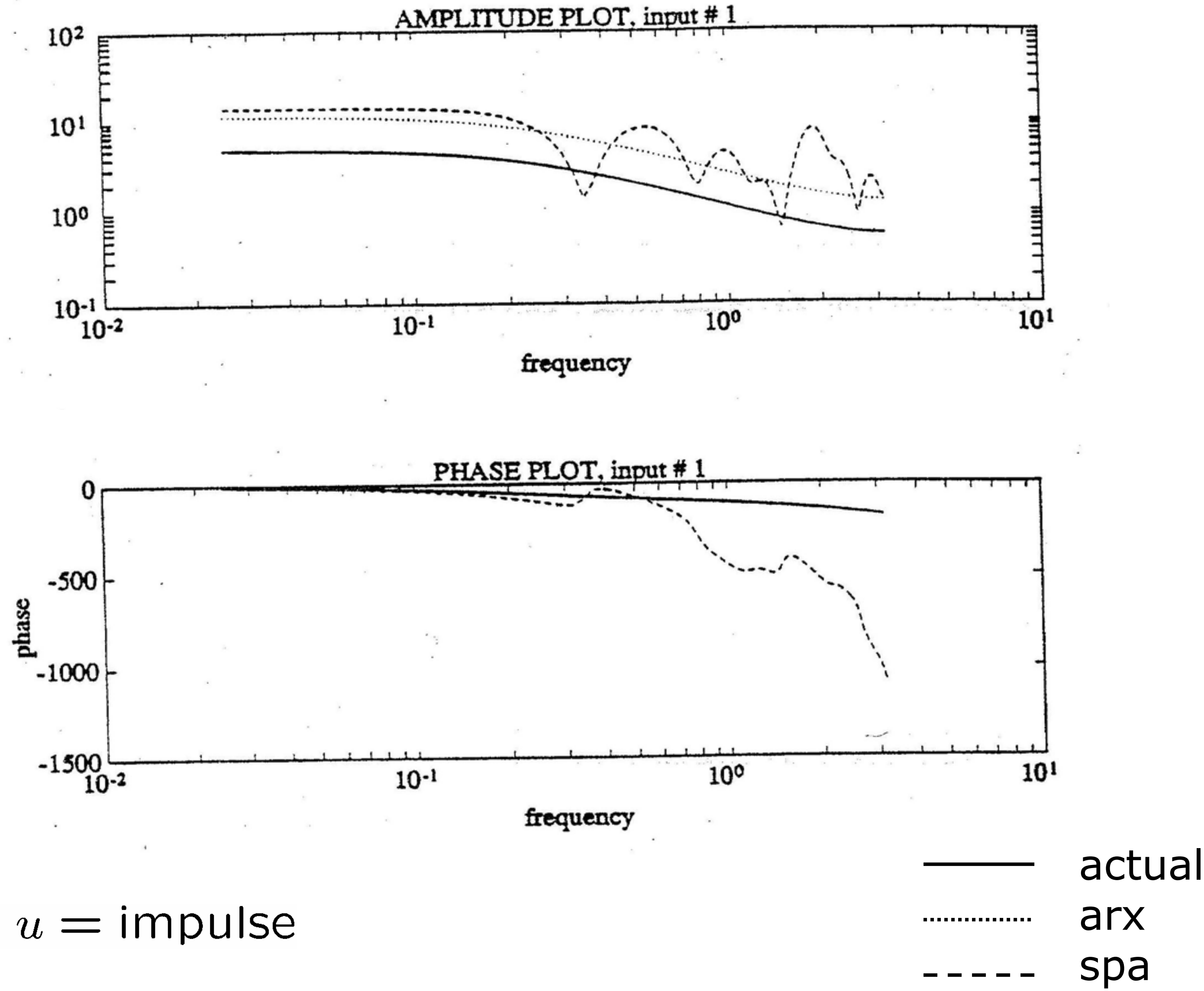
A Degenerate Input

- $u(t) = \text{impulse}$ $N = 1000$ data points

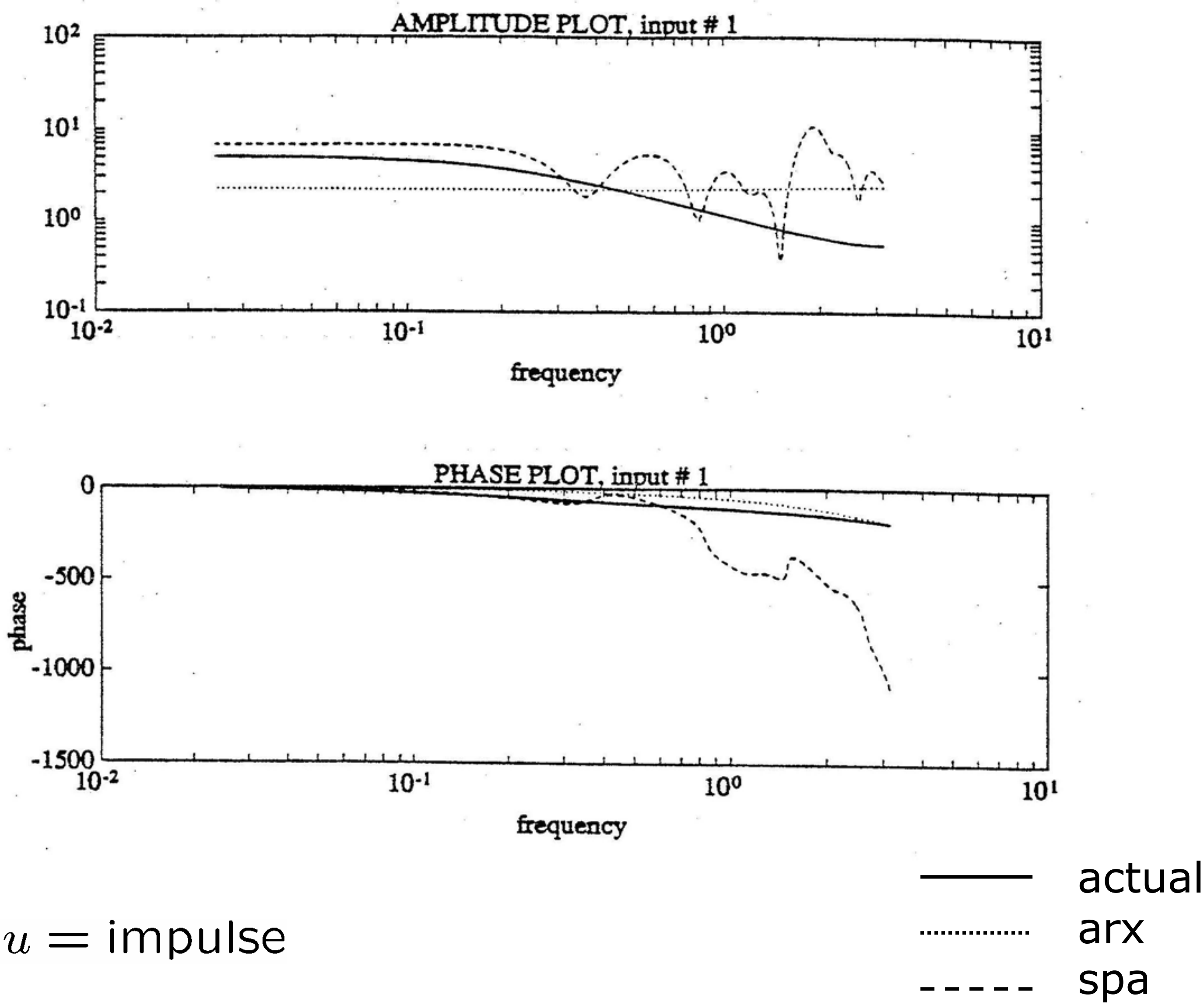
- Results from the parameter estimation:

Parameter	True value	δ_1	δ_2
a	-0.8	-0.803	0.049
b	1	2.309	2.29

- Frequency plots (with non-parametric estimates)



$\delta_1 : u =$ impulse



$\delta_2 : u =$ impulse

Theoretical Analysis

- Work it out for δ_1

$$\begin{pmatrix} \sum y^2(t-1) & -\sum y(t-1)u(t-1) \\ -\sum y(t-1)u(t-1) & \sum u^2(t-1) \end{pmatrix} \hat{\theta} = \begin{pmatrix} -\sum y(t)y(t-1) \\ \sum y(t)u(t-1) \end{pmatrix}$$

- Define

$$R_o = \frac{1}{N}\sum_{t=1}^{N} y^2(t-1) \qquad R_1 = \frac{1}{N}\sum_{t=1}^{N} y(t)y(t-1)$$

- $\hat{\theta}_N = \begin{pmatrix} NR_o & -y(1)\sigma \\ -y(1)\sigma & \sigma^2 \end{pmatrix}^{-1} \begin{pmatrix} -NR_1 \\ y(2)\sigma \end{pmatrix}$

- As $N \to \infty, u \to 0$ and

$$R_o = \frac{\lambda^2}{1-a^2} \qquad R_1 = \frac{-a\lambda^2}{1-a^2} = -a$$

- The estimates

$$\hat{a} = a_o$$

$$\hat{b} = \frac{ay(1)+y(2)}{\sigma} = b + \frac{e(2)}{\sigma}$$

Comments

- $\widehat{a}$ is estimated accurately,

  $\widehat{b}$ depends on the particular sample function for e, in particular $e(z)$.

- Different simulations yield different results !!

- Get better results if σ is very large.

- Impulse is not a rich input!