

Problem Set 9
Spring 2004

Issued: Thursday, April 15, 2004

Due: Thursday, April 29, 2004

Reading: Course notes: For this problem set: Chapter 6, Sections 7.1 and 7.2
Next: Chapter 7

Exam #2 Reminder: Our second exam will take place **Thursday, April 22, 2004, 9am - 11am**. The exam will cover material through Lecture 16 (April 8) as well as the associated homework through Problem Set 8. You are allowed to bring two $8\frac{1}{2}'' \times 11''$ sheets of notes (both sides).

Problem 9.1

A discrete-time random process $y[n]$ is observed for $n = 1, 2, \dots, N$. There are two equally likely hypotheses for the statistical description of $y[n]$:

$$\begin{aligned} H_0 : \quad y[n] &= 1 + w[n] \\ & \qquad \qquad \qquad n = 1, 2, \dots, N \\ H_1 : \quad y[n] &= -1 + w[n] \end{aligned}$$

where $w[n]$ is a non-stationary zero-mean white Gaussian sequence, *i.e.*

$$E[w[n]w[k]] = 0 \quad \text{for } n \neq k$$

(a) Assume that

$$E[w^2[n]] = n\sigma^2, \quad n = 1, 2, \dots, N.$$

Find the minimum probability of error decision rule based on observation of $y[1], \dots, y[N]$, and determine the associated error probability in terms of $\mathcal{Q}(\cdot)$ and N , where

$$\mathcal{Q}(x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-\alpha^2/2} d\alpha.$$

Find the asymptotic detection performance, *i.e.*, compute $\lim_{N \rightarrow \infty} \Pr(\text{error})$.

(b) For this part assume that

$$E[w^2[n]] = n^2\sigma^2, \quad n = 1, 2, \dots, N.$$

Find the minimum probability of error decision rule based on observation of $y[1], \dots, y[N]$, and determine the associated error probability in terms of $\mathcal{Q}(\cdot)$ and N . What is the asymptotic performance *i.e.*, what is $\lim_{N \rightarrow \infty} \Pr(\text{error})$ in this case?

Problem 9.2

Consider the following binary hypothesis testing problem:

$$\begin{aligned}
 H_0 : \quad y(t) &= (6 + v_1)s_1(t) + v_2s_2(t) + w(t) \\
 H_1 : \quad y(t) &= v_1s_1(t) + (6 + v_2)s_2(t) + w(t)
 \end{aligned}
 \qquad 0 \leq t \leq T$$

where $s_1(t)$ and $s_2(t)$ are given, orthonormal waveforms on $[0, T]$, *i.e.*,

$$\int_0^T s_1^2(t)dt = \int_0^T s_2^2(t)dt = 1, \quad \int_0^T s_1(t)s_2(t)dt = 0.$$

Also, $w[t]$ is zero-mean white Gaussian noise with

$$E[w(t)w(\tau)] = \delta(t - \tau),$$

and v_1, v_2 are independent, zero-mean Gaussian random variables, independent of $w(t)$, with

$$E[v_1^2] = 1 \quad E[v_2^2] = 2.$$

Suppose also that the two hypotheses are equally likely.

- (a) Determine the minimum probability of error decision rule for this problem, *i.e.*, specify the required processing of $y(t)$ and the subsequent threshold test.
- (b) Determine the probability of error for the decision rule of part (a) in terms of $\mathcal{Q}(\cdot)$.

Problem 9.3 (practice)

In this problem we investigate the degradation in performance that results from using a suboptimum receiver. Specifically, consider the following binary hypothesis testing problem.

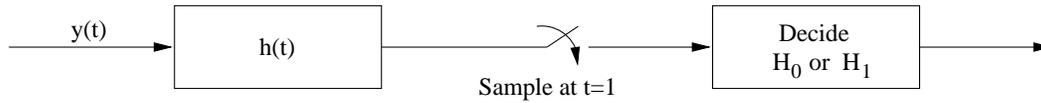
$$\begin{aligned}
 H_0 : \quad y(t) &= w(t) \\
 H_1 : \quad y(t) &= s(t) + w(t)
 \end{aligned}
 \qquad -\infty < t < \infty$$

Here $w(t)$ is white Gaussian noise with unit intensity, and

$$s(t) = \begin{cases} A & 0 \leq t \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

with A a given constant.

The general form of receivers we wish to consider is depicted below:



In this problem we are interested in the choice of the impulse response $h(t)$ and not in the specification of the threshold in the decision process.

- What is the choice for $h(t)$ in the optimal receiver?
- Suppose $h(t) = e^{-at}u(t)$. Choose the value of a to optimize the performance of this receiver.
- Compare the performance of the optimum receiver in (a) and the optimized receiver in (b). How much larger must we make A in the suboptimum design to achieve the same level of performance as in the optimum receiver?

Problem 9.4

In this problem we consider a simple phase-shift-keying (PSK) system. Specifically, suppose that we observe a signal $y(t)$ that consists of one of four equally likely signals in additive white Gaussian noise. That is, we have 4 hypotheses H_0 , H_1 , H_2 , and H_3 :

$$H_m : y(t) = A \cos\left(2\pi t + \frac{m\pi}{2}\right) + w(t), \quad 0 \leq t \leq 1$$

where A is a given constant, and $w(t)$ is zero-mean white Gaussian noise with

$$E[w(t)w(\tau)] = \delta(t - \tau).$$

- Consider the two statistics

$$\begin{aligned} \ell_1 &= \int_0^1 y(t) \cos(2\pi t) dt \\ \ell_2 &= \int_0^1 y(t) \sin(2\pi t) dt. \end{aligned}$$

Explain why knowledge of ℓ_1 and ℓ_2 is sufficient to form the optimum decision rule (*i.e.*, explain why ℓ_1 and ℓ_2 are sufficient statistics for the hypothesis testing problem). Also, determine the probability density function of

$$\begin{bmatrix} \ell_1 \\ \ell_2 \end{bmatrix}$$

under each hypothesis.

- Specify the decision regions in (ℓ_1, ℓ_2) space for the minimum probability of error decision rule.

(c) (**practice**) Show that

$$\alpha \leq \Pr [\text{error}] \leq 2\alpha$$

where

$$\alpha = \mathcal{Q} \left(\frac{d}{2} \right)$$

for some d .

Hint: Examine the decision regions determined in part (b). What is the region corresponding to an error if, say, H_0 is true? Can you represent this region as a union of two half-spaces?

Problem 9.5

Let x be an unknown parameter to be estimated based on the observation of $y(t)$ for $0 \leq t \leq T$, where

$$y(t) = \begin{cases} xs_1(t) + w(t), & x \geq 0 \\ xs_2(t) + w(t), & x < 0 \end{cases},$$

and where $w(t)$ is a zero-mean Gaussian process, with

$$E[w(t)w(\tau)] = 2\delta(t - \tau).$$

The waveforms $s_1(t)$ and $s_2(t)$ are known and satisfy

$$\int_0^T s_1^2(t) dt = \int_0^T s_2^2(t) dt = 1, \quad \int_0^T s_1(t)s_2(t) dt = 0.$$

- (a) Find the Cramér-Rao lower bound on the variance of the error of any unbiased estimate of x based on $y(t)$, $0 \leq t \leq T$.
- (b) Does an efficient estimate of x based on $y(t)$, $0 < t \leq T$, exist? Justify your answer.

Problem 9.6 (practice)

Consider the following hypothesis testing problem involving observation of $y(t)$ on $0 \leq t \leq T$ and equally likely hypotheses:

$$\begin{aligned} H_0 : & y(t) = w(t) \\ H_1 : & y(t) = x(t) + w(t) \end{aligned}$$

where $w(t)$ is zero-mean, stationary white Gaussian noise with spectral density $\sigma^2 > 0$ and $x(t)$ is a zero-mean Gaussian process. The covariance function for $x(t)$ is $K_{xx}(t, \tau)$, and has eigenvalues

$$\lambda_1 > \lambda_2 > \dots$$

and corresponding eigenfunctions

$$\phi_1(t), \phi_2(t), \dots$$

Under H_1 , $x(t)$ and $w(t)$ are statistically independent processes. Suppose we compute the following statistic

$$z = \int_0^T y(t)h(t)dt$$

via, for example, filtering, where $h(t)$ is some deterministic function satisfying

$$\int_0^T h^2(t)dt = 1.$$

We would like to make a minimum probability of error decision based on observation of the statistic z .

(a) Show that $\lambda_z = \text{var } z$ satisfies

$$\sigma^2 \leq \lambda_z \leq \sigma^2 + \int_0^T \text{var } x(t)dt$$

(b) Determine the minimum probability of error decision rule for selecting H_0 or H_1 based on observation of z . What is the corresponding probability of error?

(c) (optional) Show that the probability of error is a monotonically decreasing function of $E[z^2|H_1] - E[z^2|H_0]$, and that it is smallest when $E[z^2|H_1]$ is as large as possible.

(d) Use the result of part (c) to determine the best choice for $h(t)$, and calculate the corresponding error probability in this case.

Problem 9.7

Consider the following binary hypothesis testing problem:

$$\begin{aligned} H_0 : y(t) &= v(t) \\ H_1 : y(t) &= x(t) + v(t) \end{aligned} , \quad -\infty \leq t \leq \infty$$

where $x(t)$ is the known signal

$$x(t) = e^{-2t}u(t),$$

with $u(t)$ denoting the unit-step function, i.e.,

$$u(t) = \begin{cases} 1, & t \geq 0 \\ 0 & \text{otherwise} \end{cases}.$$

The noise $v(t)$ is a zero-mean, wide-sense stationary, Gaussian random process, with a rational spectrum $S_w(s)$, for which the associated pole-zero plot is shown in Fig. 6-1. In this figure, single poles are indicated by the symbol “ \times ”, while single zeros are denoted by the symbol “ \circ ”. Moreover,

$$S_w(s) \Big|_{s=0} = \frac{1}{4}.$$

Suppose also that the two hypotheses are equally likely.

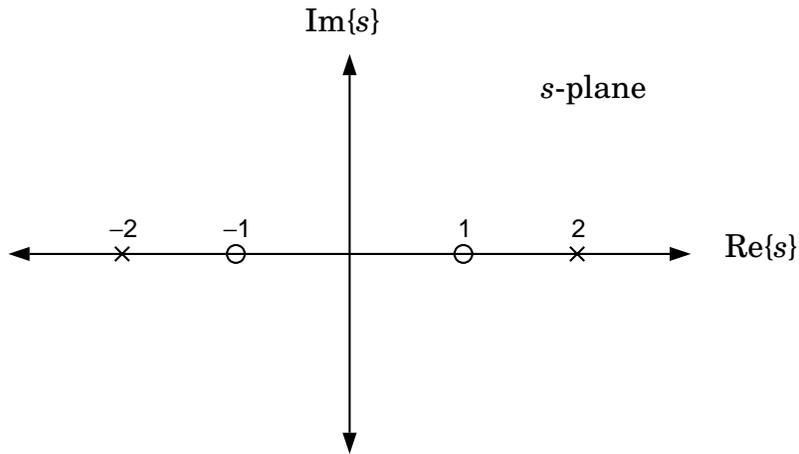


Figure 6-1

- Determine the minimum probability of error decision rule for this problem, i.e., specify the required processing of $y(t)$ (using a block diagram, for example), and the subsequent threshold test.
- Determine the probability of error for the decision rule of part (a) in terms of $\mathcal{Q}(\cdot)$, where

$$\mathcal{Q}(x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-\alpha^2/2} d\alpha.$$

Problem 9.8 (practice)

Consider the following binary hypothesis testing problem

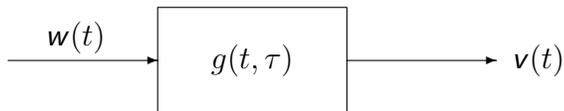
$$\begin{aligned} H_0 : y(t) &= w(t) \\ H_1 : y(t) &= s(t) + w(t) \end{aligned}, \quad 0 \leq t \leq T$$

where $w(t)$ is zero-mean, Gaussian, with

$$K_{ww}(t, \tau) = \sum_n \lambda_n \phi_n(t) \phi_n(\tau), \quad 0 \leq t, \tau \leq T$$

where $\{\phi_n(t)\}$ is a complete orthonormal basis.

- (a) Specify the time-varying, linear whitening filter



so that

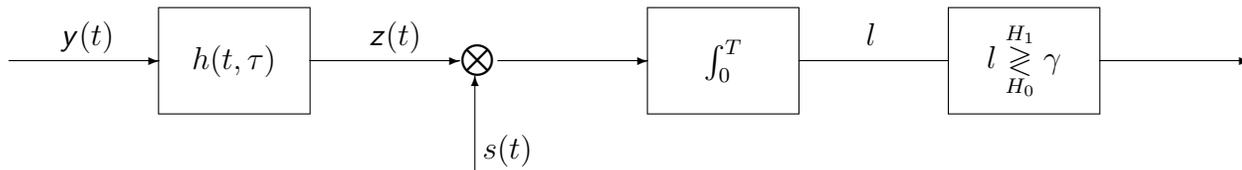
$$v(t) = \int_0^T g(t, \tau) w(\tau) d\tau$$

has as its covariance function

$$K_{vv}(t, \tau) = \delta(t - \tau)$$

That is, find $g(t, \tau)$ (a power series expansion will do).

- (b) One form for the optimal detector for this problem is depicted below



Here we aren't concerned with the threshold. Rather, we are interested in the signal processing part of this picture. Specifically, determine the filter kernel $h(t, \tau)$, where

$$z(t) = \int_0^T h(t, \tau) y(\tau) d\tau$$

As in part (a), a power series expansion for $h(t, \tau)$ will do.

- (c) Compute

$$d^2 = \frac{(E[l|H_1] - E[l|H_0])^2}{\text{Var}(l)}$$

in terms of the $\{\lambda_n\}$ and the generalized Fourier coefficients of $s(t)$

- (d) Suppose $\lambda_k = 0$ for some k . What can happen in this case?