Massachusetts Institute of Technology
Department of Electrical Engineering and Computer Science

6.432 Stochastic Processes, Detection and Estimation

Problem Set 8

Spring 2004

Issued: Thursday, April 8, 2004 **Due:** Thursday, April 15, 2004

Reading: For this problem set: Chapter 5, Sections 6.1 and 6.3
Next: Chapter 6, Sections 7.1 and 7.2

Exam #2 Reminder: Our second exam will take place **Thursday, April 22, 2004, 9am - 11am.** The exam will cover material through Lecture 16 (April 8) as well as the associated homework through Problem Set 8.
You are allowed to bring two $8\frac{1}{2}'' \times 11''$ sheets of notes (both sides).
Note that there will be **no** lecture on April 22.

Problem 8.1
Consider the continuous-time process

$$x(t) = \sum_{i=1}^{N} x_i s_i(t), \qquad 0 \le t \le T$$

where the x_i are zero-mean and jointly Gaussian with

$$E[x_i x_j] = \mu_{ij},$$

and the $s_i(t)$ are linearly independent, with

$$\int_0^T s_n(t)\, s_m(t)\, dt = \rho_{mn}.$$

(a) Calculate $K_{xx}(t, \tau)$.

(b) Is $K_{xx}(t, \tau)$ positive definite? Justify your answer.

(c) Find a matrix $\mathbf{H}$ whose eigenvalues are precisely the nonzero eigenvalues of the process $x(t)$. Determine an expression for the elements of $\mathbf{H}$ in terms of μ_{ij} and ρ_{mn}.

Hint: let

$$\phi(t) = \sum_{i=1}^{N} b_i s_i(t)$$

and introduce the vector notation

$$\mathbf{b} = [b_1\ b_2\ \cdots\ b_N]^T.$$

Show that if $\phi(t)$ is an eigenfunction of the process $x(t)$ with eigenvalue λ, then

$$\mathbf{H}\mathbf{b} = \lambda \mathbf{b}.$$

Also explain why the eigenfunctions associated with nonzero eigenvalues of $x(t)$ must have this form.

Problem 8.2
Consider a zero-mean, continuous-time process $x(t)$ with

$$K_{xx}(t,\tau) = P e^{-\alpha|t-\tau|}$$

where P, $\alpha > 0$. We would like to consider the Karhunen-Loève expansion of $x(t)$ over the time interval $[-T, T]$.

(a) Write the integral equation that an eigenfunction $\phi(t)$ and the associated eigenvalue λ must satisfy.

(b) Derive a differential equation for $\phi(t)$ from the integral equation determined in part (a).

(c) Show that $\lambda = 0$ and $\lambda = 2P/\alpha$ are not eigenvalues.

Problem 8.3
Consider the random process

$$z(t) = x(t) + v(t)$$

where $x(t)$ and $v(t)$ are independent, zero-mean (real) processes, with

$$K_{vv}(t,\tau) = \sigma_v^2 \delta(t-\tau),$$

and

$$K_{xx}(t,\tau) = \sum_{i=1}^{\infty} \lambda_i \phi_i(t) \phi_i(\tau)$$

where the $\{\phi_i(t)\}$ are real and form a complete orthonormal set in $[0, T]$. Consider

$$y = \int_0^T z(t)\, g(t)\, dt$$

where $g(t)$ is any real deterministic function. Let $\{g_i\}_{i=1}^{\infty}$ denote the coefficients of $g(t)$ in the orthonormal expansion in terms of the basis $\{\phi_i(t)\}_{i=1}^{\infty}$.

(a) Find an expression for $E\left[y^2\right]$ in terms of $g(t)$, $K_{xx}(t,\tau)$ and σ_v^2.

(b) Express $E\left[y^2\right]$ in terms of λ_i, σ_v^2, and g_i.

(c) Suppose that $g(t)$ is constrained so that

$$\int_0^T g(t)\, w(t)\, dt = 1$$

where $w(t)$ is a specified weighting function. Find the function $g(t)$ that minimizes $E\left[y^2\right]$, and the associated minimum value of $E\left[y^2\right]$.

Hint: You may find the orthonormal expansion of $w(t)$ in terms of the basis $\{\phi_i(t)\}$ useful.

Problem 8.4

Let $x(t)$ be a zero-mean wide-sense stationary stochastic process with spectrum

$$S_{xx}(j\omega) = \begin{cases} 1 + \cos(10\,\omega), & |\omega| \leq \pi/10 \\ 0, & |\omega| > \pi/10 \end{cases}$$

where ω is in rad/sec. Consider the C/D system depicted in Fig. 3-1. T denotes the sampling period, i.e., $y[n] = x(nT)$.

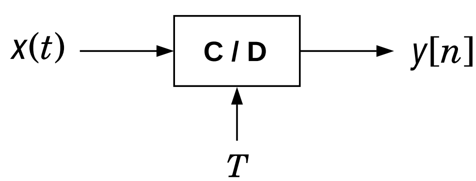


Figure 3-1

(a) Assume that the sampling period is $T = 10$ sec. Determine $S_{yy}(e^{j\omega})$ and $K_{yy}[n]$. Is it possible to reconstruct $x(t)$ from $y[n]$ in the mean-square sense? Show how the reconstruction can be achieved.

(b) Assume that the sampling period is $T = 20$ sec. Determine $S_{yy}(e^{j\omega})$ and $K_{yy}[n]$.

(c) (practice) Continue to assume that $T = 20$ sec. Show that no linear function of the $y[n]$'s can reconstruct $x(t)$ in the mean-square sense, except for $x(t)$ at the sample times $t = nT$. (*Hint:* Consider properties of the optimal linear estimate of $x(t)$ based on a finite number of the $y[n]$'s.)

Problem 8.5 (practice)

Let $x(t)$ be a zero-mean, wide-sense stationary random process with known autocorrelation function $R_{xx}(t)$. Suppose that

$$R_{xx}(nT) = R_{xx}(0)\delta[n]$$

for a given T. Let $y[n]$ be a discrete random process formed by sampling $x(t)$ with period T, *i.e.*,

$$y[n] = x(nT).$$

We wish to estimate $x(t)$ from $y[n]$ using an interpolation filter $h(t)$. That is,

$$\hat{x}(t) = \sum_{n=-\infty}^{\infty} y[n]h(t - nT).$$

Suppose $h(t)$ is constrained to be zero outside the interval $[0, MT)$, where M is a given integer. Find the function $h(t)$ that minimizes

$$J = E\left[\int_{-L}^{L} (\hat{x}(t) - x(t))^2 \, dt\right]$$

for arbitrarily large L. For the optimum $h(t)$, what is $E\left[(\hat{x}(t) - x(t))^2\right]$?
Hint: Try to pose the problem as a linear least squares estimation problem.

Problem 8.6
Suppose $x(t)$ is a random process defined as follows

$$x(t) = z[n], \qquad n < t \leq n+1, \qquad n = \ldots, -1, 0, 1, 2, \ldots$$

where $z[n]$ is a zero-mean wide-sense stationary sequence with

$$K_{zz}[k] = \begin{cases} 3, & k = 0 \\ 1, & k = \pm 1 \\ 0, & \text{otherwise} \end{cases}.$$

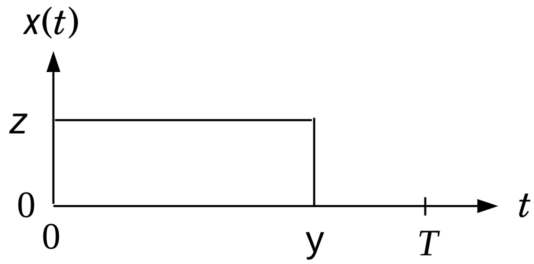
(a) Determine the covariance function $K_{xx}(t, s)$ for $0 < s \leq t < 2$.

(b) Construct a Karhunen-Loève expansion for $x(t)$ over the interval $0 < t < 2$, *i.e.*, determine the eigenvalues and eigenfunctions of the covariance function corresponding to this interval.

Problem 8.7
A random process $x(t)$ is defined as follows on the interval $0 < t < T$:

$$x(t) = \begin{cases} z, & t < y \\ 0, & t \geq y \end{cases},$$

where z is a zero-mean, unit-variance random variable and y is a random variable that is independent of z. A typical sample path is sketched below.



The covariance function for $\mathsf{x}(t)$ is given by

$$K_{\mathsf{xx}}(t,s) = 1 - \frac{\max(t,s)}{T}, \qquad 0 < s,t < T.$$

(a) Determine and make a fully labelled sketch of $p_{\mathsf{y}}(y)$.

(b) Determine a differential equation satisfied by the eigenfunctions of the process.

(c) Determine a finite upper bound on the largest eigenvalue for the process, i.e., determine M such that

$$\lambda_n \le \lambda_{\max} \le M < \infty$$

(d) Is it possible to express $\mathsf{x}(t)$ on $0 < t < T$ in the form

$$\mathsf{x}(t) = \sum_{n=1}^{N} \mathsf{x}_n \phi_n(t),$$

where the x_n are uncorrelated random variables and the functions $\phi_n(t)$ are an orthonormal set of functions with N finite? Explain.