Massachusetts Institute of Technology
Department of Electrical Engineering and Computer Science

6.432 Stochastic Processes, Detection and Estimation

Problem Set 6
Spring 2004

Issued: Tuesday, March 16, 2004 **Due:** Thursday, April 1, 2004

Reading: This problem set: Sections 4.0-4.5 and Section 4.7 except 4.7.5
Next: Sections 4.3-4.6, 4.A, 4.B, Chapter 5

Final Exam is on May 19, 2004, from 9:00am to 12:00 noon
You are allowed to bring three $8\frac{1}{2}'' \times 11''$ sheets of notes (both sides). If you have a conflict with this time and need to schedule an alternate time, you must see Prof. Willsky by April 6, 2004 at the absolute latest.

Problem 6.1
Consider the estimation of a nonrandom but unknown parameter x from an observation of the form $y = x + w$ where w is a random variable. Two different scenarios are considered in parts (a) and (b) below.

(a) Suppose w is a zero-mean Laplacian random variable. *i.e.,*

$$p_w(w) = \frac{\alpha}{2} e^{-\alpha|w|}$$

for some $\alpha > 0$. Does an unbiased estimate of x exist? Explain. Does an efficent estimate of x exist? If so, determine $\hat{x}_{\text{eff}}(y)$. If not, explain.

Hint: $\displaystyle\int_{-\infty}^{\infty} w^2 \frac{\alpha}{2} e^{-\alpha|w|}\, dw = \frac{2}{\alpha^2}.$

(b) Suppose w is a zero-mean random variable with probability density $p_w(w) > 0$ depicted in Figure 6-1. Does an efficient estimate of x exist? If so, determine $\hat{x}_{\text{eff}}(y)$. If not, explain.

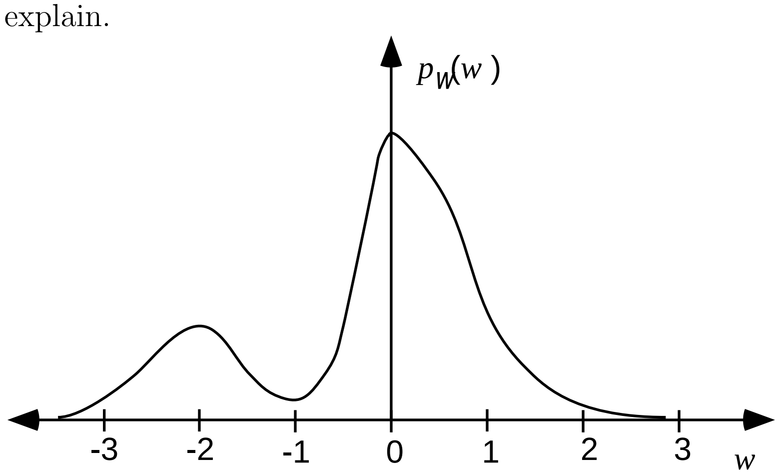

Figure 6-1

Problem 6.2

Suppose x is an unknown parameter and we have N observations of the form

$$y_k = \begin{cases} x + w_k, & x \geq 0 \\ 2x + w_k, & x < 0 \end{cases}, \qquad k = 1, 2, \ldots, N$$

where the w_k are independent and identically distributed Gaussian random variables with zero mean and variance σ^2.

(a) Determine and make a fully-labelled sketch, as a function of x, of the Cramer-Rao bound on the error variance of unbiased estimates of x.

(b) Does an efficient estimator for x exist? If so, determine $\hat{x}_{eff}(y_1, y_2, \ldots, y_N)$. If not, explain.

(c) Determine $\hat{x}_{ML}(y_1, y_2, \ldots, y_N)$, the maximum likelihood estimate for x based on $y_1, y_2, \ldots, y_N$.

(d) Is the ML estimator consistent? Explain.

Problem 6.3

A random process $x(t)$ is defined as follows:

$$x(t) = \begin{cases} a & t \geq \Theta \\ b & t < \Theta \end{cases}$$

where a, b, and Θ are statistically independent unit-variance Gaussian random variables, with means

$$\begin{aligned} E[a] &= 1 \\ E[b] &= -1 \\ E[\Theta] &= 0. \end{aligned}$$

A typical sample function is depicted in Fig. 1-1.

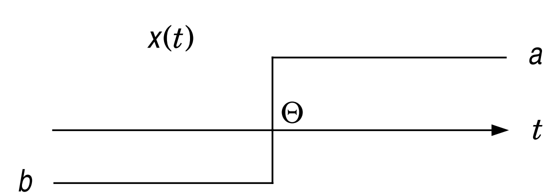

Figure 1-1

Answer each of the following questions concerning this random process, clearly justifying your answer in each case.

(a) Is $\mathsf{x}(t)$ a Gaussian random process?

(b) Is $\mathsf{x}(t)$ strict-sense stationary?

(c) Is $\mathsf{x}(t)$ a Markov process?

(d) Is $\mathsf{x}(t)$ an independent-increments process?

Problem 6.4

(a) Let $\mathsf{x}_1(t)$ be a random telegraph wave. Specifically, let $\mathsf{N}(t)$ be a Poisson counting process with

$$\Pr\left[\mathsf{N}(t) = k\right] = \frac{(\lambda t)^k e^{-\lambda t}}{k!}.$$

Let $\mathsf{x}_1(0) = +1$ with probability 1/2 and $\mathsf{x}_1(0) = -1$ with probability 1/2, and define

$$\mathsf{x}_1(t) = \begin{cases} \mathsf{x}_1(0), & N(t) \text{ even} \\ -\mathsf{x}_1(0), & N(t) \text{ odd} \end{cases}.$$

Sketch a typical sample function of $\mathsf{x}_1(t)$. Find $m_{\mathsf{x}_1}(t)$, $K_{\mathsf{x}_1\mathsf{x}_1}(t,s)$, $p_{\mathsf{x}_1(t)}(x)$, and $p_{\mathsf{x}_1(t)|\mathsf{x}_1(s)}(x_t|x_s)$. You may find that the following sums are useful:

$$\sum_{k=0}^{\infty} \frac{\alpha^k}{k!} = e^{\alpha}, \qquad \sum_{\substack{k=0 \\ k \text{ even}}}^{\infty} \frac{\alpha^k}{k!} = \cosh(\alpha), \qquad \sum_{\substack{k=0 \\ k \text{ odd}}}^{\infty} \frac{\alpha^k}{k!} = \sinh(\alpha).$$

(b) Let $\mathsf{x}_2(t)$ be a Gaussian random process with

$$\begin{aligned} m_{\mathsf{x}_2}(t) &= 0, \\ K_{\mathsf{x}_2\mathsf{x}_2}(t,s) &= e^{-2\lambda|t-s|}. \end{aligned}$$

Find $p_{\mathsf{x}_2(t)}(x)$ and $p_{\mathsf{x}_2(t)|\mathsf{x}_2(s)}(x_t|x_s)$. Sketch a typical sample function of $\mathsf{x}_2(t)$. Show that $\mathsf{x}_2(t)$ is not an independent increments process.

(c) Let

$$\mathsf{x}_3(t) = \sqrt{2}\cos(2\pi \mathsf{f} t + \Phi)$$

where f and Φ are statistically independent random variables with

$$\begin{aligned} p_{\Phi}(\Phi) &= (2\pi)^{-1}, & 0 \le \Phi \le 2\pi \\ p_{\mathsf{f}}(f) &= \frac{4\lambda}{4\lambda^2 + 4\pi^2 f^2}, & -\infty < f < \infty. \end{aligned}$$

Sketch a typical sample function of $\mathsf{x}_3(t)$. Find $m_{\mathsf{x}_3}(t)$ and $K_{\mathsf{x}_3\mathsf{x}_3}(t,s)$. An integral you may find useful is

$$\int_{-\infty}^{\infty} \left[\frac{4\lambda}{4\lambda^2 + 4\pi^2 f^2}\right] \cos(2\pi f t)\, df = e^{-2\lambda|t|}.$$

(d) Let $x(t)$ and $y(t)$ be two zero-mean random processes with covariance functions $K_{xx}(t,s)$ and $K_{yy}(t,s)$, respectively. Suppose that

$$K_{xx}(t,s) = K_{yy}(t,s) \qquad \forall t, s.$$

Does $E\left[(x(t) - y(t))^2\right] = 0$ for all t? Explain.

Problem 6.5 (practice)
Let

$$x(t) = \cos(2\pi f_0 t + \Phi)$$

where $f_0 > 0$ is a constant and Φ is a random variable with

$$p_\Phi(\Phi) = \frac{1}{4}\left[\delta(\Phi) + \delta\left(\Phi - \frac{\pi}{2}\right) + \delta(\Phi - \pi) + \delta\left(\Phi - \frac{3\pi}{2}\right)\right].$$

(a) Find $m_x(t)$ and $K_{xx}(t,s)$. Is $x(t)$ wide-sense stationary?

(b) Find $p_{x(t)}(x)$. Is $x(t)$ strict-sense stationary?

Problem 6.6
Let $N_1(t)$ and $N_2(t)$ be two statistically independent homogeneous Poisson counting processes with rates λ_1 and λ_2, respectively. We define a new process $z(t)$ by the relation

$$z(t) = y\, N_1(t)\, (-1)^{N_2(t)}, \qquad t \geq 0,$$

where y is a random variable that is statistically independent of the processes $N_1(t)$ and $N_2(t)$ and has a density

$$p_y(y) = \frac{1}{2}(\delta(y+1) + \delta(y-1)).$$

(a) Sketch a typical sample function of $z(t)$ when $\lambda_2 = \lambda_1/4$.

(b) Find $m_z(t)$ and $K_{zz}(t,s)$ for $t, s \geq 0$.

(c) For $t > s > 0$ find $\hat{z}(t)$, the linear least squares estimate of $z(t)$ based on observation of $z(s)$. Is $z(t)$ an independent-increments process? Explain briefly.

Problem 6.7
Let $x(t)$ be a non-white, zero-mean, Gaussian, wide-sense stationary, Markov random process; and let

$$y(t) = x(|t|).$$

Answer each of the following questions concerning this new random process $y(t)$, clearly justifying your answer in each case.

(a) Is $y(t)$ a Gaussian random process?

(b) Is $y(t)$ strict-sense stationary?

(c) Is $y(t)$ a Markov process?

(d) Is $y(t)$ an independent-increments process?