Massachusetts Institute of Technology
Department of Electrical Engineering and Computer Science

6.432 Stochastic Processes, Detection and Estimation

Problem Set 5
Spring 2004

Issued: Thursday, March 4, 2004 **Due:** Tuesday, March 16, 2004

Reading: This problem set: Sections 3.3.3 - 3.3.6, 3.4
Next: Sections 4.0 - 4.5 and Section 4.7 except 4.7.5

Exam #1 Reminder: Our first quiz will take place **Thursday, March 11, 9am - 11am**. The exam will cover material through Lecture 8 (March 2) as well as the associated Problem Sets 1 through 4.
You are allowed to bring one $8\frac{1}{2}'' \times 11''$ sheet of notes (both sides).

Problem 5.1

(a) Let

$$p_y(y;x) = \begin{cases} x & \text{if } 0 \le y \le 1/x \\ 0 & \text{otherwise} \end{cases},$$

for $x > 0$. Show that there exist no unbiased estimators $\hat{x}(y)$ for x. (Note that because only $x > 0$ are possible values, an unbiased estimator need only be unbiased for $x > 0$ rather than all x.)

(b) Suppose instead that

$$p_y(y;x) = \begin{cases} \frac{1}{x} & \text{if } 0 \le y \le x \\ 0 & \text{otherwise} \end{cases},$$

for $x > 0$. Does a minimum-variance unbiased estimator for x based on y exist? If your answer is yes determine $\hat{x}_{MVU}(y)$. If your answer is no, explain.

Problem 5.2 (practice)
The data $x[n] = Ar^n + w[n]$ for $n = 0, \ldots, N-1$ are observed. The random variables $w[0], \ldots, w[N-1]$ are i.i.d. Gaussian random variables with zero mean and variance σ^2. Find the Cramér-Rao bound for A. Does an efficient estimator exist? If so, what is it and what is its variance? For what values of r is it consistent?

Problem 5.3 (practice)
Let x be an unknown nonrandom scalar parameter, and suppose that the N-dimensional random vector $\mathbf{y}$ represents some observed data, with mean and covariance given by

$$E[\mathbf{y}] = \mathbf{c}x + \mathbf{d}, \qquad \mathbf{\Lambda_y}(x) = \mathbf{\Lambda}$$

where $\mathbf{c}$, $\mathbf{d}$, $\mathbf{\Lambda}$ are all known.

(a) An estimator $\hat{x}(\mathbf{y})$ is linear if

$$\hat{x}(\mathbf{y}) = \mathbf{a}^T\mathbf{y} + b$$

for some $\mathbf{a}$, b.

Find the unbiased linear estimator $\hat{x}(\mathbf{y})$ with the minimum variance.
(*Hint*: Lagrange multipliers.)
Remark: In the estimation literature, this is frequently (and somewhat ambiguously) called the "best linear unbiased estimator" (BLUE).

(b) What is the variance of your estimator in part (a)?

Problem 5.4
Suppose, for i=1,2

$$y_i = x + w_i$$

where x is an unknown constant, and where w_1 and w_2 are statistically independent, zero-mean Gaussian random variables with

$$\begin{aligned} \operatorname{var} w_1 &= 1 \\ \operatorname{var} w_2 &= \begin{cases} 1 & x \geq 0 \\ 2 & x < 0 \end{cases}. \end{aligned}$$

(a) Calculate the Cramér-Rao bound for unbiased estimators of x based on observation of

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}.$$

(b) Show that a minimum variance unbiased estimator $\hat{x}_{\mathrm{MVU}}(\mathbf{y})$ does not exist.

Hint: Consider the estimators

$$\begin{aligned} \hat{x}_1 &= \frac{1}{2}y_1 + \frac{1}{2}y_2 \\ \hat{x}_2 &= \frac{2}{3}y_1 + \frac{1}{3}y_2. \end{aligned}$$

Problem 5.5
The purpose of this problem is to determine the (unknown) probability of heads when flipping a particular coin.

(a) Suppose that the coin is flipped M times in succession, each toss statistically independent of all others, each with (unknown) probability p of heads. Let n be the number of heads that is observed. Find the maximum likelihood (ML) estimate of p based on knowledge of n.

(b) Evaluate the bias and the mean-square error of the ML estimate.

(c) Is the ML estimate efficient? Is it consistent? Briefly explain.

Problem 5.6 (practice)

Suppose we observe a random N-dimensional vector $\mathbf{y}$, whose components are independent, identically-distributed Gaussian random variables, each with mean x_1 and variance x_2.

(a) Suppose x_1 is unknown but x_2 is known. Does an efficient estimate exist? Find the maximum likelihood estimate of x_1 based on observation of $\mathbf{y}$. Evaluate the bias and the mean-square error for this estimate.

(b) Suppose x_1 is known but x_2 is unknown. Does an efficient estimate exist? Find the maximum likelihood estimate of x_2 based on observation of $\mathbf{y}$. Evaluate the bias and the mean-square error for this estimate.

(c) Suppose both x_1 and x_2 are unknown. Does an efficient estimate exist? Find $\hat{\mathbf{x}}_{\mathrm{ML}}(\mathbf{y})$, the maximum likelihood estimate of

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

based on observation of $\mathbf{y}$. Evaluate $E\left[\hat{\mathbf{x}}_{\mathrm{ML}}(\mathbf{y})\right]$ and $E\left[(\hat{x}_{1_{\mathrm{ML}}}(\mathbf{y}) - x_1)^2\right]$. Compare with your results from parts (a) and (b). Evaluate $E\left[(\hat{x}_{2_{\mathrm{ML}}}(\mathbf{y}) - x_2)^2\right]$ assuming $N = 2$. Compare with your result from part (b).