Massachusetts Institute of Technology
Department of Electrical Engineering and Computer Science

6.432 Stochastic Processes, Detection and Estimation

Problem Set 4
Spring 2004

Issued: Thursday, February 26, 2004 **Due:** Thursday, March 4, 2004

Reading: This problem set: Sections 1.7, 3.2.5, 3.3.1, 3.3.2
Next: Sections 3.3.3 - 3.3.6, 3.4

Exam #1 Reminder: Our first quiz will take place **Thursday, March 11, 9 am - 11 am.** The exam will cover material through Lecture 8 (March 6), as well as the associated Problem Sets 1 through 4.
You are allowed to bring one $8\frac{1}{2}'' \times 11''$ sheet of notes (both sides). Note that there will be **no** lecture on March 11.

Problem 4.1
Suppose w, z are scalar random variables, and that

$$p_z(z) = \begin{cases} 1/2 & |z| < 1 \\ 0 & \text{otherwise} \end{cases}.$$

You are told that the Bayes least-squares estimate of w given an observation z is

$$\hat{w}_{\text{BLS}} = -\frac{1}{2}\operatorname{sgn} z = \begin{cases} -1/2 & z \geq 0 \\ 1/2 & \text{otherwise} \end{cases},$$

and the associated mean-square estimation error is $\lambda_{\text{BLS}} = 1/12$. However, you would prefer to use the following ad-hoc estimator

$$\hat{w}_{\text{AH}} = -z.$$

(a) Is it possible to determine $b(\hat{w}_{\text{AH}}) = E[w - \hat{w}_{\text{AH}}]$, the bias of your new estimator, from the information given? If your answer is no, briefly explain your reasoning. If your answer is yes, calculate $b(\hat{w}_{\text{AH}})$.

(b) Is it possible to determine $\lambda_{\text{AH}} = E[(w - \hat{w}_{\text{AH}})^2]$, the mean-square estimation error obtained using this estimator, from the information given? If your answer is no, briefly explain your reasoning. If your answer is yes, calculate λ_{AH}.

Problem 4.2
Let x, y, z be zero-mean, unit-variance random variables which satisfy

$$\text{var}[x + y + z] = 0$$

Find the covariance matrix of $(x, y, z)^T$; i.e., find the matrix

$$\begin{bmatrix} E[x^2] & E[xy] & E[xz] \\ E[yx] & E[y^2] & E[yz] \\ E[zx] & E[zy] & E[z^2] \end{bmatrix}$$

(Hint: Use vector space ideas.)

Problem 4.3
Suppose $y \sim N(0, \sigma^2)$ and that the Bayes least-squares estimate of a related random variable x based on y is

$$\hat{x}_{\text{BLS}}(y) = y^2.$$

(a) Could x and y be jointly Gaussian? Explain briefly.

(b) Determine $\hat{x}_{\text{LLS}}(y)$, the linear least-squares estimate of x based on y (*i.e.*, your estimator should be of the form $\hat{x}_{\text{LLS}}(y) = \alpha y + \beta$).

(c) If the error variance of your estimator in (b) is $\lambda_{\text{LLS}} = 3\sigma^4$, determine λ_{BLS}.

Note: Recall that if v is a Gaussian random variable with zero mean and variance σ_v^2, then

$$E[v^n] = \begin{cases} (\sigma_v)^n \cdot 1 \cdot 3 \cdot 5 \cdots (n-1), & n \text{ even} \\ 0, & n \text{ odd} \end{cases}.$$

Problem 4.4
Consider the communication system shown below. The message x is a $N(0, \sigma_x^2)$ random variable. The transmitter output is hx, and the receiver input is

$$y = hx + v,$$

where v is an $N(0, r)$ random variable that is statistically independent of x.

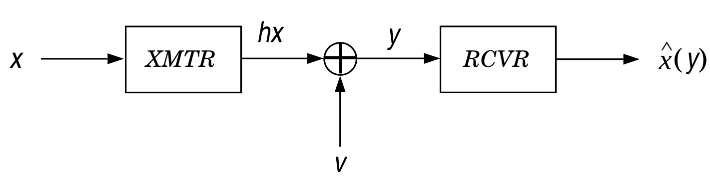

Figure 3-1

Suppose the transmitter is subject to intermittent failure, i.e., h is a random variable taking the values 0 and 1 with probabilities $1 - p$ and p, respectively. Assume h,x, and v are mutually independent.

(a) Find $\hat{x}_{\text{LLS}}(y)$, the linear least-squares estimate of x based on observation of y, and λ_{LLS}, its resulting mean-square estimation error.

(b) Prove that

$$E[x|y=y] = \sum_{i=0}^{1} \Pr[h=i|y=y]\; E[x|y=y, h=i].$$

(c) Find $\hat{x}_{\mathrm{BLS}}(y)$, the Bayes least-squares estimate of x based on observation of y.

Problem 4.5

Let

$$\mathbf{z} = \begin{bmatrix} w \\ v \end{bmatrix}$$

be a 2-dimensional Gaussian random vector with mean $\mathbf{m_z}$ and covariance matrix $\mathbf{\Lambda_z}$

$$\mathbf{m_z} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \qquad \mathbf{\Lambda_z} = \begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix}.$$

Let x be a Gaussian random variable with mean $m_x = 2$ and variance $\lambda_x = 8$. Assume x and $\mathbf{z}$ are statistically independent. The random variable y is related to x and $\mathbf{z}$ as follows

$$y = (2+w)x + v.$$

(a) Find $\hat{x}_{LLS}(y)$, the linear least-squares estimate of x based on observation of y.

(b) Determine $\lambda_{LLS} = E[(x - \hat{x}_L(y))^2]$, the resulting mean-square estimation error.

Problem 4.6

Let x and y be random variables such that the random variable x is exponential, and, conditioned on knowledge of x, y is exponentially distributed with parameter x, i.e.,

$$\begin{aligned} p_x(x) &= \frac{1}{a} e^{-x/a} u(x) \\ p_{y|x}(y|x) &= x e^{-xy} u(y) \end{aligned}$$

where

$$u(t) = \begin{cases} 1 & \text{if } t \geq 0 \\ 0 & \text{otherwise} \end{cases}.$$

(a) Determine $\hat{x}_{\mathrm{BLS}}(y)$, $\lambda_{x|y}(y) = E\left[[\hat{x}_{\mathrm{BLS}}(y) - x]^2 \,|y=y\right]$, and $\lambda_{\mathrm{BLS}} = E\left[[\hat{x}_{\mathrm{BLS}}(y) - x]^2\right]$.

(b) Determine $\hat{x}_{\mathrm{MAP}}(y)$, the MAP estimate of x based on observation of y. Determine the bias and error variance for this estimator. (Recall that the bias is $E[\hat{x}_{\mathrm{MAP}}(y) - x]$.)

(c) Find $\hat{x}_{\text{LLS}}(y)$, the linear least-squares estimate of x based on observation of y, and λ_{LLS}, the resulting mean-square estimation error.

Problem 4.7 (practice)

(a) Let $\mathbf{d}$ be a random vector. Let x, y, w be random variables to be estimated based on $\mathbf{d}$. Furthermore, $w = x + y$.

(i) Show that

$$\hat{w}_{\text{BLS}}(\mathbf{d}) = \hat{x}_{\text{BLS}}(\mathbf{d}) + \hat{y}_{\text{BLS}}(\mathbf{d}).$$

(ii) Show that

$$\hat{w}_{\text{LLS}}(\mathbf{d}) = \hat{x}_{\text{LLS}}(\mathbf{d}) + \hat{y}_{\text{LLS}}(\mathbf{d}).$$

(b) Let x be a zero-mean scalar random variable. Let $y[0], \ldots, y[N]$ be a sequence of zero-mean scalar random variables. At each time k, the value of $y[k]$ is revealed to the estimator. At each time k, the estimator must construct the linear least-squares estimate of x based on all the data it has seen up to that point in time. That is, at time k, the estimator must construct

$$\hat{x}[k] \triangleq \hat{x}_{\text{LLS}}(y[0], \ldots, y[k]).$$

Show that

$$\hat{x}[k] = \hat{x}[k-1] + K[k](y[k] - \hat{y}[k|k-1]), \qquad k = 1, \ldots, N$$

where $\hat{y}[k|k-1]$ is the linear least-squares estimate of $y[k]$ based on $y[0], \ldots, y[k-1]$, and

$$K[k] = \frac{E\left[x(y[k] - \hat{y}[k|k-1])\right]}{E\left[(y[k] - \hat{y}[k|k-1])^2\right]}.$$

(Hint: Use orthogonality.)

(c) Let $y[n] = x + w[n]$ for $n = 0, \ldots, N$, where $x, w[0], \ldots, w[N]$ are zero-mean mutually independent random variables. Also, for $n = 0, \ldots, N$, $\text{var}\, w[n] = \sigma^2$. Letting $\hat{x}[n]$ be defined as in (b), show that

$$\begin{aligned} \hat{x}[k] &= \hat{x}[k-1] + K[k](y[k] - \hat{x}[k-1]) \\ K[k] &= \frac{E\left[(x - \hat{x}[k-1])^2\right]}{E\left[(x - \hat{x}[k-1])^2\right] + \sigma^2}. \end{aligned}$$

Why might this recursive form of the estimator be convenient?