Massachusetts Institute of Technology
Department of Electrical Engineering and Computer Science

6.432 Stochastic Processes, Detection and Estimation

Problem Set 10
Spring 2004

Issued: Thursday, April 29, 2004 **Due:** Thursday, May 6, 2004

Final Exam: Our final will take place on **May 19, 2004, from 9:00am to 12:00 noon.**
You are allowed to bring three $8\frac{1}{2}'' \times 11''$ sheets of notes (both sides).

Reading: For this problem set: Chapter 7
Next: Section 4.8, Addenda to Chapters 6 and 7

Problem 10.1
Let

$$y(t) = x(t) + v(t)$$

where $x(t)$ and $v(t)$ are uncorrelated, zero-mean processes, with

$$S_{xx}(s) = \frac{3}{1-s^2}$$

$$S_{vv}(s) = \frac{5}{9-s^2}$$

(a) Find the noncausal Wiener filter extimating $x(t)$ based on $y(t)$. Also find the corresponding mean-square estimation error.

(b) Find the causal and causally invertible whitening filter for $y(t)$. You will find that whitening $y(t)$ requires differentiation.

(c) Find the causal Wiener filter for estimating $x(t)$. You should find that the overall filter doesn't involve any differentiation. Also, find the associated mean-square estimation error.

Problem 10.2
Consider the system depicted in Fig. 2-1 (on the next page), where $w(t)$ and $v(t)$ are independent, zero-mean noise processes with

$$E\left[w(t)w(\tau)\right] = \delta(t-\tau)$$
$$E\left[v(t)v(\tau)\right] = \frac{1}{5}\delta(t-\tau) \ .$$

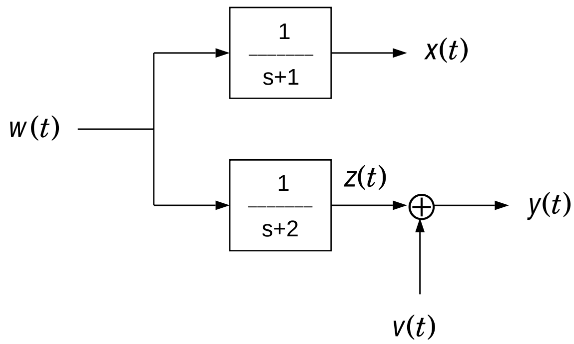

Figure 2-1

(a) Determine the noncausal Wiener filter for estimation $x(t)$ based on observation of $y(t)$. Also, compute the associated mean-square error.

(b) Determine the causal Wiener filter for estimating $x(t)$ based on observation of $y(t)$. Again, compute the associated mean-square error.

Problem 10.3 (practice)
Let $x[n]$ be a discrete-time process generated as illustrated in Fig. 3-1, where $w[n]$ is a zero-mean, wide-sense stationary white noise process with variance q, and where $G(z)$ is the system function of the stable LTI system (depicted in Fig. 3-1), described by the difference equation

$$x[n+1] = \alpha x[n] + w[n], \qquad |\alpha| < 1$$

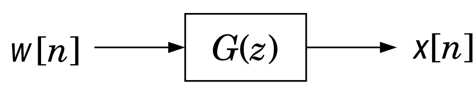

Figure 3-1

(a) What is $S_{xx}(z)$?

(b) Suppose we observe $y[n] = x[n] + v[n]$, where $v[n]$ is a zero-mean white noise process with variance r, and $v[n]$ is independent of $w[n]$. Find the system function of the non-causal system as depicted in Fig. 3-2 that minimizes the mean-square estimation error $E\left[(x[n] - \hat{x}[n])^2\right]$.

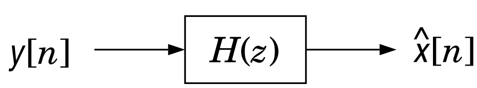

Figure 3-2

(c) Let $q = 3$, $r = 4$, and $\alpha = 1/2$.

(i) Show that $H(z)$ has the following form

$$H(z) = -\frac{\beta z}{(z-\gamma)(z-\gamma^{-1})}$$

by finding the two numbers β and γ, with $|\gamma| < 1$.

(ii) Write $H(z) = H_1(z)H_2(z)$, where $H_1(z)$ is the system function of a stable, causal system, and $H_2(z)$ is the system function of a stable, anticausal system (where for an anticausal system the output depends only on future and present values of the input). Thus, the optimum system $H(z)$ can be realized as a cascade of a causal and an anticausal system, both of which are stable.

(iii) (practice) Write $H(z) = H_3(z)+H_4(z)$, where $H_3(z)$ is the system function of a stable, causal system, and $H_4(z)$ is the system function of a stable, anticausal system. This shows that $H(z)$ can be realized as a parallel connection of a causal system and an anticausal system. Also use this representation to determine $h[n]$, the impulse response corresponding to $H(z)$.

(d) Using the numerical values given in part (c), determine the numerical value of the mean-square error $E\left[(\mathsf{x}[n] - \hat{\mathsf{x}}[n])^2\right]$. You may leave your answer in terms of β and γ.

(e) Determine the causal Wiener filter for estimating $\mathsf{x}[n]$ based on $\mathsf{y}[n]$. Again, use the numerical values given in part (c).

Problem 10.4

Let $\mathsf{y}(t)$ be a zero-mean stochastic process with

$$S_{\mathsf{yy}}(s) = \frac{1}{(1-s^2)^2}.$$

Let T be a (given) positive number. Find the optimal Wiener filter for estimating $\mathsf{y}(t+T)$ given $\mathsf{y}(\tau)$ for $-\infty < \tau \leq t$, and compute the associated mean-square estimation error.

Problem 10.5

Let $\mathsf{x}[n]$ be a wide-sense stationary, zero-mean, unit-variance, discrete-time random process. Let $\hat{\mathsf{x}}[n+1]$ denote the linear least- squares estimate of $\mathsf{x}[n+1]$ based on $\mathsf{x}[n]$ and $\mathsf{x}[n-1]$. This estimate can be generated by the linear time-invariant system in Fig. 5-1.

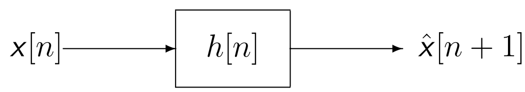

Figure 5-1

The unit-sample response is

$$h[n] = \begin{cases} 1/3 & n = 0, 1 \\ 0 & \text{otherwise} \end{cases}$$

and is depicted in Fig. 5-2.

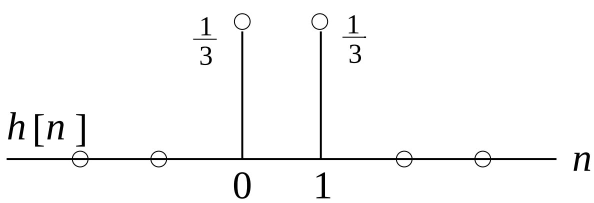

Figure 5-2

(a) Determine as many samples of the auto-correlation sequence

$$R_{xx}[k] = E\left[x[n]x[n-k]\right]$$

as possible from the information given. If it is not possible to determine any of the samples, explain.

(b) Is it possible to determine $\lambda_L = E\left[(\hat{x}[n+1] - x[n+1])^2\right]$? If your answer is yes, compute λ_L. If your answer is no, explain.

Problem 10.6 (practice)
Let $x(t)$ be a zero-mean WSS random process with autocorrelation function

$$R_{xx}(t) = e^{-\lambda|t|}.$$

Let $y[n] = x(nT)$ be a discrete-time process formed by sampling $x(t)$.

(a) Find the optimal noncausal interpolation filter for recovering $x(t)$ from $y[n]$. That is, find $h(t)$ so as to minimize

$$J(t) = E\left[(x(t) - \hat{x}(t))^2\right]$$

for all t, where

$$\hat{x}(t) = \sum_{n=-\infty}^{\infty} y[n]h(t - nT).$$

Find the resulting $J(t)$.

(b) Find the optimal causal interpolation filter for recovering $x(t)$ from $y[n]$. That is, find $h(t)$ so as to minimize

$$J(t) = E\left[(x(t) - \hat{x}(t))^2\right]$$

for all t, where

$$\hat{x}(t) = \sum_{n=-\infty}^{m} y[n]h(t - nT)$$

for $mT \le t < (m+1)T$. Find the resulting $J(t)$.

Problem 10.7

Consider the following binary hypothesis testing problem:

$$\begin{aligned} H_0 &: y[n] = v[n] \\ & \qquad\qquad\qquad\qquad n = 0, 1, 2 \\ H_1 &: y[n] = \delta[n] + v[n] \end{aligned}$$

where $\delta[n]$ is the unit impulse and $v[n]$ is a zero-mean, Gaussian noise process with

$$E[v^2[n]] = 4 \qquad E[v[n]v[n-1]] = 2 \qquad E[v[n]v[n-2]] = 0$$

for all n. Since this is a correlated noise process, we need to whiten it. Do this by the innovations method, i.e. by Gram-Schmidt orthogonalization.

(a) Find the processor depicted in Fig. 7-1 so that

$$\begin{aligned} w[0] &= v[0] \\ w[1] &= v[1] - \hat{v}[1|0] \\ w[2] &= v[2] - \hat{v}[2|1] \end{aligned}$$

where $\hat{v}[n|k]$ is the Bayes least-squares estimate of $v[n]$ based on $v[0], \cdots, v[k]$. Determine $E[w[n]]$, $E[w^2[n]]$, $E[w[n]w[n-1]]$, and $E[w[n]w[n-2]]$.

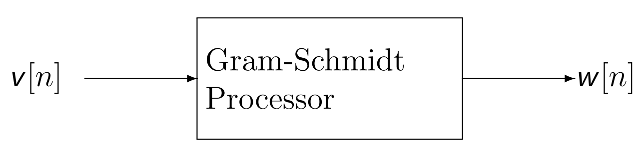

Figure 7-1

(b) The processor in part (a) can be thought of as a linear system operating on a sequence of inputs. Use this processor as depicted in Fig. 7-2 to find the minimum probability of error decision rule for this hypothesis testing problem. Assume H_0 and H_1 are equally likely.

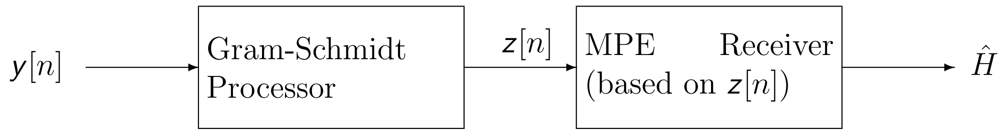

Figure 7-2