Massachusetts Institute of Technology
Department of Electrical Engineering and Computer Science

6.432 Stochastic Processes, Detection and Estimation

Problem Set 1
Spring 2004

Issued: Tuesday, February 3, 2004 **Due:** Tuesday, February 10, 2004

Reading: For this problem set: Chapter 1 of course notes, through Section 1.6, Appendix 1.A
Next: Chapter 2, through Section 2.5.1

Problem 1.1
A random variable x has probability distribution function

$$P_{\mathsf{x}}(x) = [1 - \exp(-2x)]\, u(x)$$

where $u(\cdot)$ is the unit-step function.

(a) Calculate the following probabilities:

$$\Pr[\mathsf{x} \leq 1], \qquad \Pr[\mathsf{x} \geq 2], \qquad \Pr[\mathsf{x} = 2].$$

(b) Find $p_{\mathsf{x}}(x)$, the probability density function for x.

(c) Let y be a random variable obtained from x as follows:

$$\mathsf{y} = \begin{cases} 0 & \mathsf{x} < 2 \\ 1 & \mathsf{x} \geq 2 \end{cases}.$$

Find $p_{\mathsf{y}}(y)$, the probability density function for y.

Problem 1.2
Let x and y be independent identically distributed random variables with common density function

$$p(\alpha) = \begin{cases} 1 & 0 \leq \alpha \leq 1 \\ 0 & \text{otherwise} \end{cases}.$$

Let $\mathsf{s} = \mathsf{x} + \mathsf{y}$.

(a) Find and sketch $p_{\mathsf{s}}(s)$.

(b) Find and sketch $p_{\mathsf{x}|\mathsf{s}}(x|s)$ vs. x with s viewed as a known parameter.

(c) The conditional mean of x given $\mathsf{s} = s$ is

$$E[\mathsf{x} \mid \mathsf{s} = s] = \int_{-\infty}^{+\infty} x\, p_{\mathsf{x}|\mathsf{s}}(x|s)\, dx.$$

Find $E[\mathsf{x} \mid \mathsf{s} = 0.5]$.

(d) The conditional mean of x given s (s viewed as a random variable) is

$$m_{\mathsf{x}|\mathsf{s}} = E[\mathsf{x}|\mathsf{s}] = \int_{-\infty}^{+\infty} x\, p_{\mathsf{x}|\mathsf{s}}(x|\mathsf{s})\, dx.$$

Since $m_{\mathsf{x}|\mathsf{s}}$ is a function of the random variable s, it too is a random variable. Find the density function for $m_{\mathsf{x}|\mathsf{s}}$.

Problem 1.3

Let x be a random variable with probability density function $p_{\mathsf{x}}(x)$. The Fourier transform of $p_{\mathsf{x}}(x)$, denoted

$$M_{\mathsf{x}}(jv) = \int_{-\infty}^{+\infty} p_{\mathsf{x}}(x)\, e^{jvx}\, dx,$$

is called the characteristic function of x.

(a) Find $p_{\mathsf{x}}(x)$ for

$$M_{\mathsf{x}}(jv) = \operatorname{sinc}\left(\frac{v}{\pi}\right) = \frac{\sin(v)}{v}.$$

(b) For

$$M_{\mathsf{x}}(jv) = \frac{\lambda^2}{\lambda^2 + v^2}$$

find m_{x} and σ_{x}^2 *without* first computing $p_{\mathsf{x}}(x)$.

Problem 1.4

A dart is thrown at random at a wall. Let (x, y) denote the Cartesian coordinates of the point in the wall pierced by the dart. Suppose that x and y are statistically independent Gaussian random variables, each with mean zero and variance σ^2, i.e.,

$$p_{\mathsf{x}}(\alpha) = p_{\mathsf{y}}(\alpha) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{\alpha^2}{2\sigma^2}\right).$$

(a) Find the probability that the dart will fall within the σ-radius circle centered at the point $(0, 0)$.

(b) Find the probability that the dart will hit in the first quadrant ($\mathsf{x} \geq 0$, $\mathsf{y} \geq 0$).

(c) Find the conditional probability that the dart will fall within the σ-radius circle centered at $(0,0)$ given that the dart hits in the first quadrant.

(d) Let $\mathsf{r} = (\mathsf{x}^2 + \mathsf{y}^2)^{1/2}$, and $\Theta = \tan^{-1}(\mathsf{y}/\mathsf{x})$ be the polar coordinates associated with (x, y). Find $\Pr[0 \le \mathsf{r} \le r,\ 0 \le \Theta \le \Theta]$ and obtain $p_{\mathsf{r},\Theta}(r, \Theta)$.
This observation leads to a widely used algorithm for generating Gaussian random variables.

Problem 1.5
Consider the following 3×3 matrices:

$$\mathbf{A} = \begin{bmatrix} 10 & 3 & 1 \\ 2 & 5 & 0 \\ 1 & 0 & 2 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 10 & 5 & 2 \\ 5 & 3 & 3 \\ 2 & 3 & 2 \end{bmatrix} \quad \mathbf{C} = \begin{bmatrix} 10 & 5 & 2 \\ 5 & -3 & 3 \\ 2 & 3 & 2 \end{bmatrix} \quad \mathbf{D} = \begin{bmatrix} 10 & 5 & 2 \\ -5 & 3 & 1 \\ -2 & -1 & 2 \end{bmatrix}$$

$$\mathbf{E} = \begin{bmatrix} 10 & -5 & 2 \\ -5 & 3 & -1 \\ 2 & -1 & 2 \end{bmatrix} \quad \mathbf{F} = \begin{bmatrix} 10 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix} \quad \mathbf{G} = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$$

Your answers to the following questions may consist of more than one of the above matrices or none of them. Justify your answers.

(a) Which of the above could be the covariance matrix of some random vector?

(b) Which of the above could be the cross-covariance matrix of two random vectors?

(c) Which of the above could be the covariance matrix of a random vector in which one component is a linear combination of the other two components?

(d) Which of the above *could be* the the covariance matrix of a vector with statistically independent components? Must a random vector with such a covariance matrix have statistically independent components?

Problem 1.6

(a) Consider the random variables x, y whose joint density function is given by (see Fig. 6-1)

$$p_{\mathsf{x},\mathsf{y}}(x, y) = \begin{cases} 2 & \text{if } x, y \ge 0 \text{and } x + y \le 1 \\ 0 & \text{otherwise} \end{cases}.$$

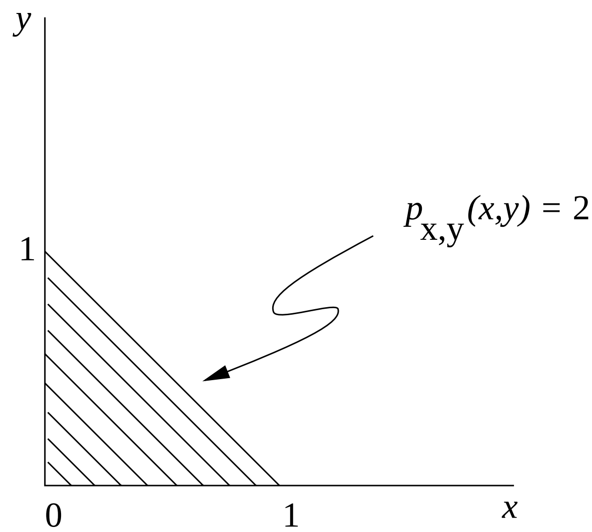

Figure 6-1

(i) Compute the covariance matrix

$$\Lambda = \begin{bmatrix} \lambda_x & \lambda_{xy} \\ \lambda_{xy} & \lambda_y \end{bmatrix}.$$

(ii) Knowledge of y generally gives us information about the random variable x (and vice versa). We want to estimate x based on knowledge of y. In particular, we want to estimate x as an affine function of y, *i.e.*,

$$\hat{x} = \hat{x}(y) = ay + b,$$

where a and b are constants. Select a and b so that the expected mean-square error between x and its estimate $\hat{x}$, *i.e.*,

$$E[(\hat{x} - x)^2],$$

is minimized.

(iii) Provide a labeled sketch of $p_{x|y}(x|y)$ for an arbitrary value of y between 0 and 1.

(iv) Compute $E[x|y]$ and

$$\Lambda_{x|y}(y) = E[(x - E[x|y])^2|y = y]$$

(v) Compute $E[x]$ using iterated expectations.

(b) Consider the random variables x, y whose joint density function is given by (see Fig. 6-2)

$$p_{x,y}(x,y) = \begin{cases} 1 & \text{if } 0 \leq x, y \leq 1 \\ 0 & \text{otherwise} \end{cases}.$$

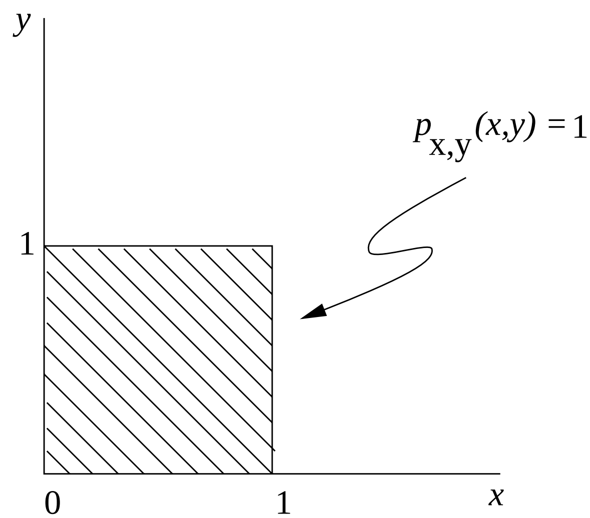

Figure 6-2

Repeat steps (i) - (v) of part (a), and compare your results to those on part (a).