Massachusetts Institute of Technology
Department of Electrical Engineering and Computer Science

6.341: Discrete-Time Signal Processing

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Lecture 4

DT Processing of CT Signals
& CT Processing of DT Signals: Fractional Delay

Reading: Sections 4.1 - 4.5 in Oppenheim, Schafer & Buck (OSB).

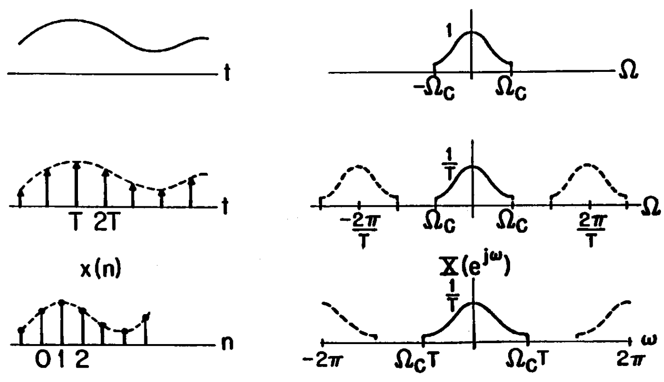
A typical discrete-time system used to process a continuous-time signal is shown in OSB Figure 4.15. T is the sampling/reconstruction period for the C/D and the D/C converters. When the input signal is bandlimited, the effective system in Figure 4.15 (b) is equivalent to the continuous-time system shown in Figure 4.15 (a).

Ideal C/D Converter

The relationship between the input and output signals in an ideal C/D converter as depicted in OSB Figure 4.1 is:

$$\begin{aligned} \text{Time Domain:} \quad & x[n] &=& \; x_c(nT) \\ \text{Frequency Domain:} \quad & X(e^{j\omega}) &=& \; \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\left(j\left(\frac{\omega}{T} - \frac{2\pi k}{T}\right)\right), \end{aligned}$$

where $X(e^{j\omega})$ and $X_c(j\Omega)$ are the DTFT and CTFT of $x[n]$, $x_c(t)$. These relationships are illustrated in the figures below. See Section 4.2 of OSB for their derivations and more detailed descriptions of C/D converters.



Time/Frequency domain representation of C/D conversion

In the frequency domain, a C/D converter can be thought of in three stages: periodic repetition of the spectrum at 2π, normalization of frequency by $\frac{1}{T}$, and scaling of the amplitude by $\frac{1}{T}$. The CT frequency Ω is related to the DT frequency ω by $\Omega = \frac{\omega}{T}$. If $x_c(t)$ is not bandlimited to $-\frac{\Omega_s}{2} < \Omega_{max} < \frac{\Omega_s}{2}$, repetition at 2π will cause aliasing. Note that the C/D converter here is an idealized sampler, representing the mathematical operations of sampling with a periodic impulse train followed by conversion from impulse train to a discrete-time sequence. It does *not* represent physical sampling circuits. In practice, the ideal C/D converter is approximated by analog-to-digital (A/D) converters, which will also quantize the input signal to a finite number of amplitude levels. OSB figure 4.2 illustrates C/D conversion of the same signal at two different sampling rates.

Ideal D/C Converter

An ideal D/C converter is depicted in OSB Figure 4.10. As discussed in Section 4.3 of OSB, the input/output relationship of this system is

$$\text{Time Domain:} \qquad x_r(t) = \sum_n x[n]\text{sinc}\left(\frac{t-nT}{T}\right)$$

$$\text{Frequency Domain:} \qquad X_r(j\Omega) = \begin{cases} TX(e^{j\omega}) & \Omega \le \frac{\pi}{T} \\ 0 & \text{otherwise} \end{cases}$$

A D/C reconstructs the continuous signal by bandlimited interpolation. In the frequency domain, this is equivalent to ideal low-pass filtering of the scaled spectrum. That is

$$X_r(j\Omega) = H_r(j\Omega)X(e^{j\Omega T}) \ ,$$

where

$$H_r(j\Omega) = \begin{cases} T & |\Omega| \le \frac{\pi}{T} \\ 0 & \text{otherwise} \ . \end{cases}$$

See OSB Figure 4.8 for the block diagram of such an ideal bandlimited filtering system, and OSB Figure 4.9 for the time-domain representation of bandlimited interpolation. Note that $x[n] = x_r(nT)$ is *not a definition* of a D/C converter, because it does not specify the values of $x_r(t)$ for time instances $t \neq nT$.

In the frequency domain, a D/C can be interpreted as carrying out three operations in reverse of C/D conversion: bandlimiting to the base period $-\frac{\Omega_r}{2} \le \omega \le \frac{\Omega_r}{2} = \frac{\pi}{T}$, normalization of frequency by T, and scaling of the amplitude by T. Similar to C/D converters, a D/C converter is also ideal, representing the mathematical operations of conversion from sequence to an impulse train, followed by idealized reconstruction.

DT Processing of CT Signals

OSB Figure 4.15 shows the cascade of a C/D converter, a discrete-time system, followed by a D/C converter. In this system, if $x_c(t)$ is bandlimited, i.e. $X_c(j\Omega) = 0$ for $|\Omega| \le \frac{\pi}{T}$, the overall system is linear and time invariant, and equivalent to a continuous time LTI system:

$$H_{\text{eff}}(j\Omega) = H(j\Omega) \qquad \text{for } |\Omega| \le \frac{\pi}{T} \ .$$

For example, Figure 4.13 shows the Fourier transform of signals through the overall system when the input is bandlimited. The discrete-time system in this case is an ideal lowpass filter with cutoff frequency ω_c.

Example:

$H_c(j\omega)$	:	Lowpass filter, cutoff frequency = $\pi/4$
$1/T$	:	$20kHz$
$H_{eff}(j\Omega)$	:	Lowpass filter, cutoff frequency = ?

CT Processing of DT Signals: Fractional Delay

In addition to using discrete-time systems for processing continuous-time signals, the complementary situation is also possible, where a continuous time system is preceded by a D/C converter and followed by a C/D converter to process discrete-time signals, as depicted in OSB Figure 4.16. Section 4.5 of OSB analyzes such a system in detail. Given a CT system $H_c(j\Omega)$, the overall DT system behaves as:

$$H(e^{j\omega}) = H_c(j\frac{\omega}{T}) \ , \qquad |\omega| < \pi \ .$$

Equivalently, the overall system will be the same as a given $H(e^{j\omega})$, if the continuous-time system satisfies

$$H_c(j\Omega) = H(e^{j\Omega T}) \ , \qquad |\Omega| < \pi/T \ .$$

An important example of such a system is discussed in OSB Example 4.9: consider a discrete-time system with the following frequency response:

$$H(e^{j\omega}) = e^{-j\omega\Delta} \ .$$

When Δ is an integer, this is a time delay by Δ units: $y[n] = x[n-\Delta]$. When Δ is not an integer, however, such an interpretation is incorrect, since a discrete sequence can only be shifted by integer amounts. Instead, consider choosing $H_c(j\Omega)$ in OSB Figure 4.16 to be

$$H_c(j\Omega) = H(e^{j\Omega T}) = e^{-j\Omega\Delta T} \ .$$

It then follows that

$$y_c(t) = x_c(t - \Delta T) \ .$$

With some computations, we see that the output of this system can be interpreted as bandlimited interpolation of the input discrete-time signal, followed by non-integer time delay and re-sampling. Summarizing using a direct convolution representation with $-\infty < n < \infty$:

$$y[n] = \sum_{k=\infty}^{\infty} x[k] \frac{\sin \pi(n-k-\Delta)}{\pi(n-k-\Delta)} = x[n] * \frac{\sin \pi(n-\Delta)}{\pi(n-\Delta)} = x[n] * h[n] \ .$$