

**Lecture 3**  
**Minimum-Phase and All-Pass Systems**

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**Reading:** Sections 5.5 and 5.6 in Oppenheim, Schafer & Buck (OSB).

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## All-Pass Systems

Definition of an all-pass system  $H_{AP}(z)$  is as follows:

$$|H_{AP}(e^{j\omega})| = A$$

The gain of an all-pass system is a real constant (A doesn't necessarily need to be 1).

In order to satisfy the above definition, each pole of  $H_{AP}(z)$  should be paired with a conjugate reciprocal zero, as shown in OSB Figure 5.21.

A rational all-pass system has the general form given below:

$$H_{AP}(z) = A \prod_{k=1}^p \frac{(z^{-1} - a_k^*)}{(1 - a_k z^{-1})} \quad |a_k| < 1$$

If a pole is at  $z = a_k$  then a zero is at  $z = 1/a_k^*$ , i.e. a pole at  $a_k = re^{j\theta}$  is paired with a zero at  $\frac{1}{a_k^*} = \frac{1}{r}e^{j\theta}$ . If  $h[n]$  is real, then  $a_k = a_k^*$ .

OSB Figure 5.24 shows the frequency response for an all-pass system with the pole-zero plot in OSB Figure 5.21. Note that Figure 5.24(b) shows the wrapped phase. The group delay in (c) is largest when  $\omega = \pm\pi/4$  and  $\pi$ , the points on the unit circle that are closest to the poles and zeros. The phase change is greatest around these points.

An all-pass system is always stable, since when frequency response characteristics (such as all-pass) are discussed, it is naturally assumed that the Fourier transform exists, thus stability is implied.

**Example:**

$H(z) = z^{-1}$  has a pole at the origin, and a zero at  $\infty$ , thus it is an all-pass system. In general, any rational function  $H(z)$  will have an equal number of poles and zeros (some at  $\infty$ ).

## Minimum-Phase Systems

The basic definition of a minimum-phase system is as follows:  
Stable & causal and has a stable & causal inverse

Stable and causal  $\Leftrightarrow$  All poles of  $H(z)$  are inside the unit circle.

Stable and causal inverse  $\Leftrightarrow$  All poles of  $1/H(z)$  are inside the unit circle or equivalently, all zeros of  $H(z)$  are inside the unit circle.

Thus, to have a minimum-phase system, all poles and zeros of  $H(z)$  must be inside the unit circle (no pole or zero at  $\infty$ ). Since the number of poles is always equal to the number of zeros, you have the same number of poles and zeros inside the unit circle. The system in OSB Figure 5.30(a) has four poles and four zeros inside the unit circle, and thus is minimum-phase.

### Example:

$$H(z) = z - \frac{1}{2}: \text{NOT minimum-phase (a pole at } \infty)$$

From the definitions, it is clear that an all-pass system cannot be minimum-phase.

## Spectral Factorization

Generally, several different systems can have different phase responses and yet have the same magnitude response. However, for a minimum-phase signal  $h[n]$ , the frequency response can be uniquely recovered (to within a sign change) from the magnitude alone. This also means that you cannot specify both magnitude and phase independently for a minimum-phase system.

For a real rational system:

$$|H(e^{j\omega})|^2 = H(e^{j\omega})H^*(e^{j\omega}) = H(e^{j\omega})H(e^{-j\omega}) = H(z)H(1/z)|_{z=e^{j\omega}}.$$

The following example demonstrates the process of recovering  $H(z)$  from the given  $|H(e^{j\omega})|^2$ .

### Example:

$$|H(e^{j\omega})|^2 = \frac{\frac{17}{16} - \frac{1}{2} \cos \omega}{\frac{5}{4} - \cos \omega} = \frac{\frac{17}{16} - \frac{1}{4}e^{j\omega} - \frac{1}{4}e^{-j\omega}}{\frac{5}{4} - \frac{1}{2}e^{j\omega} - \frac{1}{2}e^{-j\omega}}$$

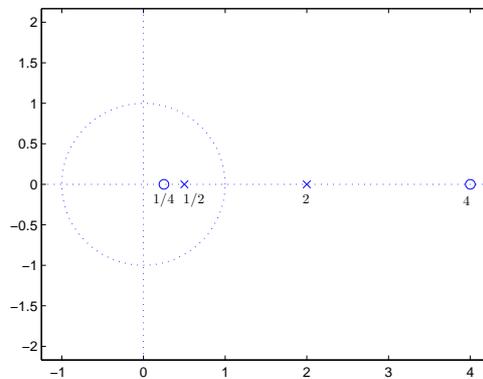
We will just replace  $e^{j\omega}$  with  $z$  (analytic continuation).

$$\implies H(z)H(1/z)|_{z=e^{j\omega}} = \frac{\frac{17}{16} - \frac{1}{4}z - \frac{1}{4}z^{-1}}{\frac{5}{4} - \frac{1}{2}z - \frac{1}{2}z^{-1}}|_{z=e^{j\omega}}$$

Let  $G(z) = H(z)H(1/z)$ , then we need to factor  $G(z)$ .

Zeros of  $G(z)$  :  $z = 4, 1/4$

Poles of  $G(z)$  :  $z = 2, 1/2$



Since  $H(z)$  is minimum-phase, we know that  $H(z)$  must have all its poles and zeros inside the unit circle, thus it has a pole at  $1/2$  and a zero at  $1/4$ .

$$H(z) = A \frac{1 - \frac{1}{4}z^{-1}}{1 - \frac{1}{2}z^{-1}} \quad H(1/z) = A \frac{1 - \frac{1}{4}z}{1 - \frac{1}{2}z}$$

$$H(z)H(1/z)|_{z=1(\omega=0)} = A^2 \frac{(1 - \frac{1}{4})^2}{(1 - \frac{1}{2})^2} = A^2 \frac{(\frac{3}{4})^2}{(\frac{1}{2})^2} = A^2 (\frac{3}{2})^2$$

From the given  $|H(e^{j\omega})|^2$ ,

$$|H(e^{j\omega})|^2|_{\omega=0} = \frac{\frac{17}{16} - \frac{1}{2}}{\frac{5}{4} - 1} = \frac{\frac{9}{16}}{\frac{1}{4}} = \frac{9}{4} \implies A = 1$$

This problem of recovering frequency response from the magnitude response is commonly called spectral factorization.

## Maximum-Phase Systems

The basic definition of a maximum-phase system is as follows:

Stable and anti-causal with a stable and anti-causal inverse.

⇔ All poles and zeros are outside the unit circle and ROC includes the unit circle.

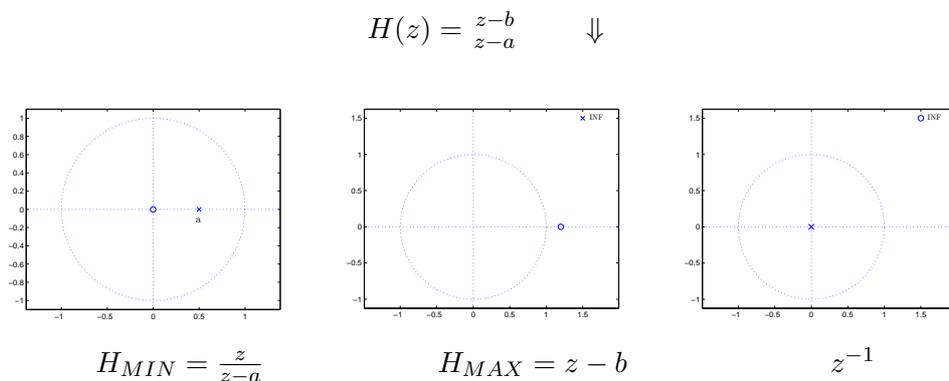
OSB Figure 5.30(b) shows the pole/zero plot of a delayed maximum-phase system. The system is stable but causal due to the 4th order pole at  $z = 0$ . The corresponding sequences associated with OSB Figure 5.30 are shown in OSB Figure 5.31. Note that  $h_b[n]$  is the flipped and delayed version of  $h_a[n]$  as  $H_b(z) = c \cdot z^{-4} \cdot H_a(1/z)$ .

A stable system  $H(z)$  can always be expressed as

$$H(z) = H_{MIN}(z) H_{MAX}(z) \cdot z^{-M}$$

for some integer  $M$ . The factor  $z^{-M}$  allows us to compensate for poles and zeros at the origin or at  $\infty$ .

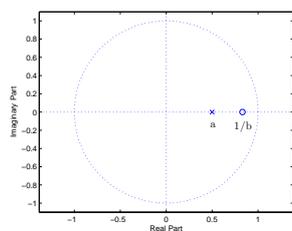
One example of the factorization is shown below.  $H_{MIN}(z)$  takes the the pole inside the unit circle and  $H_{MAX}(z)$  takes the zero outside the unit circle. In order to make the number of poles and zeros equal, we place a zero at the origin for  $H_{MIN}$ , and a pole at  $\infty$  for  $H_{MAX}$ . The factor  $z^{-1}$  compensates these zero and pole.



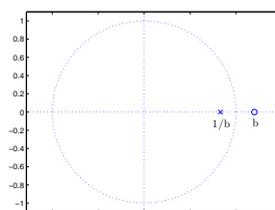
A stable system  $H(z)$  can also be expressed as

$$H(z) = H_{MIN}(z)H_{AP}(z).$$

$$H(z) = \frac{z-b}{z-a} \quad \Downarrow$$



$$H_{MIN} = \frac{z-1/b}{z-a}$$



$$H_{AP} = \frac{z-b}{z-1/b}$$

OSB problem 5.64 illustrates the importance of this concept in compensating the magnitude response of a nonminimum-phase system.

$$x[n] \longrightarrow \boxed{H(z)} \longrightarrow \boxed{H_c(z)} \longrightarrow y[n]$$

$$H(z) = H_{MIN}(z) H_{AP}(z)$$

Since  $H_{MIN}(z)$  has a stable and causal inverse, we can define a stable and causal system  $H_c(z)$  as follows:

$$H_c(z) = \frac{1}{H_{MIN}(z)}$$

With this compensating system, the magnitude of the overall frequency response is unity.

$$|Y(e^{j\omega})| = |H(e^{j\omega}) H_c(e^{j\omega})| |X(e^{j\omega})| = |H_{AP}(e^{j\omega})| |X(e^{j\omega})| = |X(e^{j\omega})|$$

There are different ways of defining minimum-phase systems. Now, we will talk about what's "minimum" about minimum-phase systems.

1. Minimum-phase systems have minimum group delay.
2. Minimum-phase systems have minimum energy delay.

### Minimum Group Delay

As shown in OSB section 5.5, for a causal all-pass system,

$$\tau_{AP}(\omega) \geq 0 \quad \forall \omega.$$

The equality holds for all  $\omega$  when  $H_{AP}(z)$  is just a constant.

Consider a single factor :

$$H_{AP}(z) = \frac{z^{-1} - a_k^*}{1 - a_k z^{-1}} \quad |a_k| < 1$$

Let  $a_k = r e^{j\theta}$ , then

$$\tau(\omega) = \frac{1 - r^2}{|1 - r e^{j\theta} e^{-j\omega}|^2} > 0$$

Now, consider all casual systems with the same frequency response magnitude :  $|H(e^{j\omega})| = |H_i(e^{j\omega})|$

$$H_i(z) = H_{MIN}(z) H_{AP_i}(z)$$

If  $H_{AP_i}(z)$  is not a constant, then it increases the group delay, i.e. among all causal systems with the same frequency response magnitude, the minimum-phase one has the smallest group delay at all frequencies.

$$\tau_i(\omega) \geq \tau_{MIN}(\omega) \quad \forall \omega$$

and the equality holds iff  $H_i(z) = H_{MIN}(z)$ .

### Minimum Energy Delay

Closely related is the minimum energy delay property.

$$\sum_{m=0}^n h_{MIN}^2[m] \geq \sum_{m=0}^n h^2[m]$$

i.e. the minimum-phase impulse response is the most "front-loaded" of all the causal ones with the same frequency response magnitude. The energy accumulates faster for the minimum-phase impulse response as shown in OSB Figure 5.32. OSB Problem 5.66 proves this result.