

Lecture 2

Background Review

Phase, Group Delay, and Generalized Linear Phase

Reading: Sections 5.1, 5.3, and 5.7 in Oppenheim, Schafer & Buck (OSB).

Phase

$$x[n] \xrightarrow{\text{LTI}} \boxed{H(z)} \longrightarrow y[n]$$

The frequency response $H(e^{j\omega})$ of an LTI system $H(z)$ is evaluated on the unit circle, $|z| = 1$.

$$H(e^{j\omega}) = H(z) \big|_{z=e^{j\omega}}$$

The Fourier transforms of the input $x[n]$ and the output $y[n]$ of the system are related by

$$Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega}).$$

We can understand this input-output relationship in more detail by looking at the equation in terms of the magnitude-phase representation.

Magnitude/Phase Representation

$$H(e^{j\omega}) = H_R(e^{j\omega}) + jH_I(e^{j\omega}) = |H(e^{j\omega})|e^{j\theta_1(\omega)}, \quad \theta_1(\omega) = \angle H(e^{j\omega})$$

Example:

$$\begin{aligned} h[n] &= -\delta[n] \\ \Rightarrow |H(e^{j\omega})| &= 1, \quad \angle H(e^{j\omega}) = \pi \end{aligned}$$

In the mag/phase representation, a real-valued frequency response does not necessarily mean that the system is zero-phase.

Using this representation,

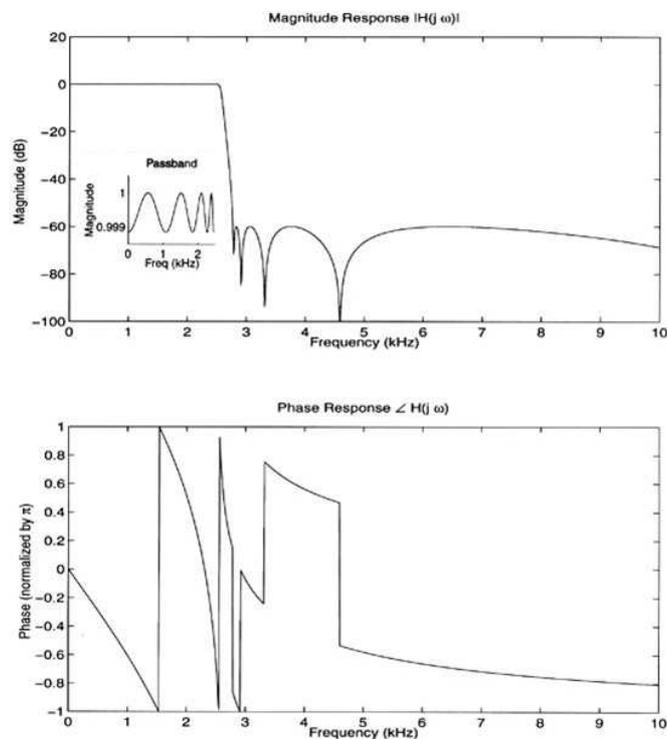
$$|Y(e^{j\omega})| = |H(e^{j\omega})||X(e^{j\omega})|$$

and

$$\angle Y(e^{j\omega}) = \angle H(e^{j\omega}) + \angle X(e^{j\omega}).$$

Thus, $|H(e^{j\omega})|$ and $\angle H(e^{j\omega})$ are commonly referred to as the gain and the phase shift of the system, respectively.

In magnitude and phase plots, as ω goes through a zero on the unit circle, the magnitude will go to zero and the phase will flip by π , as shown in the figure below.



Magnitude and phase response of an elliptic lowpass filter

If $H(e^{j\omega})$ is real but bipolar, it's often more natural to use an alternative representation to remove these jumps of π in a phase plot.

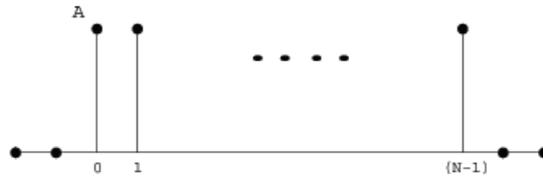
Amplitude/Phase Representation

$$H(e^{j\omega}) = A(e^{j\omega})e^{j\theta_2(\omega)}, \quad \theta_2(\omega) = \angle H(e^{j\omega})$$

$A(e^{j\omega})$ is real but not restricted to be positive, thus $\theta_2(\omega)$ is different from $\theta_1(\omega)$ above. The sign change is contained in $A(e^{j\omega})$, and the jumps of π do not exist in $\theta_2(\omega)$.

Example:

Consider $h[n]$ given in the figure below.



$$H(e^{j\omega}) = Ae^{-j\omega(N-1)/2} \frac{\sin(\omega N/2)}{\sin(\omega/2)}$$

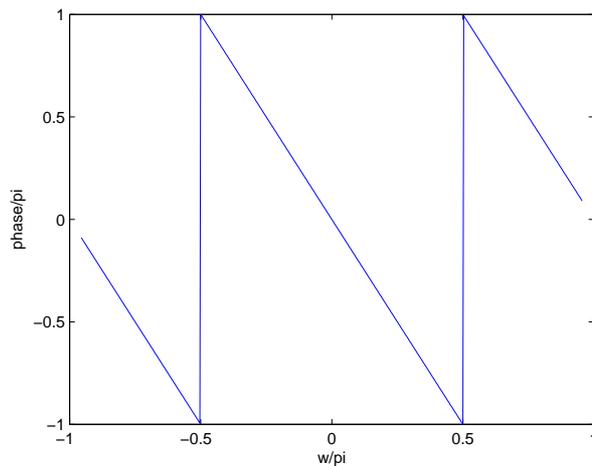
In the magnitude/phase representation, $\theta_1(\omega)$ has jumps of π at the sign change. In the amplitude/phase representation, $\theta_2(\omega) = -\omega(N-1)/2$, a straight line with the slope $-(N-1)/2$. Moreover, in this representation, phase would be the same whether A is positive or negative.

Wrapped vs. Unwrapped Phase

Example:

$$H(e^{j\omega}) = e^{-2j\omega}$$

In either the mag/phase or the amp/phase representations, $\angle H(e^{j\omega}) = -2\omega$. However, if you plot the phase in MATLAB, it will give you the wrapped phase shown below.



In general, phase is ambiguous since $e^{j(\theta+2\pi n)} = e^{j\theta}$ for any integer n . OSB Figure 5.7 shows the continuous phase (denoted as \arg) and the wrapped phase (denoted as ARG) of an LTI system.

Group Delay

Typically, it is hard to infer much from a phase plot, and a group delay plot gives more useful information.

$$\tau(\omega) = \text{grad}[H(e^{j\omega})] \equiv -\frac{d}{d\omega}[\arg H(e^{j\omega})] = -\frac{d}{d\omega}[\text{ARG } H(e^{j\omega})]$$

The last equality holds except at discontinuities of $[\text{ARG } H(e^{j\omega})]$.

In MATLAB, group delay is calculated using the Fourier transform rather than differentiation. In the amplitude/phase representation:

$$H(e^{j\omega}) = A(\omega)e^{j\theta(\omega)} \tag{1}$$

Differentiating both sides,

$$H'(e^{j\omega}) = A'(\omega)e^{j\theta(\omega)} + A(\omega)e^{j\theta(\omega)}(j\theta'(\omega))$$

and dividing by Equation 1,

$$\frac{H'(e^{j\omega})}{H(e^{j\omega})} = \frac{A'(\omega)}{A(\omega)} + j\theta'(\omega)$$

Since both $\frac{A'(\omega)}{A(\omega)}$ and $\theta'(\omega)$ are real,

$$\tau(\omega) = -\text{Im}\left[\frac{H'(e^{j\omega})}{H(e^{j\omega})}\right].$$

If $h[n] \leftrightarrow H(e^{j\omega})$, we can apply the frequency differentiation property of the Fourier transform and get

$$-jnh[n] \leftrightarrow H'(e^{j\omega}).$$

Denoting Fourier transform of $h[n]$ as $F.T.(h[n])$,

$$\tau(\omega) = -\text{Im}\left[\frac{F.T.(-jnh[n])}{F.T.(h[n])}\right]$$

and finally,

$$\tau(\omega) = \text{Re}\left[\frac{F.T.(nh[n])}{F.T.(h[n])}\right].$$

Systems with Linear Phase

Systems with constant group delay are referred to as linear phase systems. These systems are desirable when we want to minimize the distortion on the shape of a signal. The three systems given below are linear phase systems.

System 1: $h_1[n]$ is symmetric about zero.

$$h_1[n] = h_1[-n]$$

$H_1(e^{j\omega})$ is real, and phase is zero in the amplitude/phase representation. Thus, the group delay is zero, and $h_1[n]$ is linear phase.

System 2: $h_2[n]$ is symmetric about an integer n_0 , i.e. $h_1[n]$ delayed by n_0 .

$$h_2[n] = h_1[n - n_0]$$

$$H_2(e^{j\omega}) = e^{-j\omega n_0} H_1(e^{j\omega})$$

$H_1(e^{j\omega})$ is real, so the group delay of $H_2(e^{j\omega})$ is n_0 , and $h_2[n]$ is also linear phase.

Since $h_1[n]$ is even,

$$h_2[n + n_0] = h_1[n] = h_1[-n] = h_2[-n + n_0],$$

and $h_2[n]$ satisfies the following:

$$h_2[n] = h_2[2n_0 - n]$$

System 3: $h_3[n] = h_3[2\alpha - n]$, where 2α is an integer.

$$H_3(e^{j\omega}) = H_3^*(e^{j\omega})e^{-j2\alpha\omega},$$

since $h_3[n] = h_3[-(n - 2\alpha)]$. Multiplying both sides by $e^{j\omega\alpha}$,

$$e^{j\omega\alpha} H_3(e^{j\omega}) = H_3^*(e^{j\omega})e^{-j\omega\alpha} = \overline{e^{j\omega\alpha} H_3(e^{j\omega})}.$$

This implies that both sides must be real, so $H_3(e^{j\omega})$ can be expressed as

$$H_3(e^{j\omega}) = A(e^{j\omega})e^{-j\omega\alpha},$$

where $A(e^{j\omega})$ is real (possibly bipolar). Thus, the phase response of $H_3(e^{j\omega})$ is linear, and the group delay is constant.

Since $h[n]$ is only defined at integer values, we need 2α to be an integer to represent $h[2\alpha - n]$. If α is an integer, then System 3 corresponds to System 2 and $h[n]$ is symmetric about α (see OSB Figure 5.35 (a)). If 2α is an integer but α is not an integer, the point of symmetry lies between samples (see OSB Figure 5.35 (b)). When 2α is not an integer, $h[n]$ is not symmetric, but the envelop is symmetric (see OSB Figure 5.35 (c)).

Generalized Linear Phase

Consider a system which is antisymmetric about zero:

$$h[n] = -h[n]$$

This implies that $H(e^{j\omega})$ is purely imaginary, therefore in the amplitude/phase representation,

$$H(e^{j\omega}) = jI(\omega) = I(\omega)e^{j\pi/2},$$

where $I(\omega)$ is real. Thus, we see that the phase of $H(e^{j\omega})$ is $\pi/2$, and the group delay is zero.

If the system has an odd symmetric impulse response

$$h[n] = -h[2\alpha - n]$$

where 2α is an integer, then it follows that

$$H(e^{j\omega}) = -H^*(e^{j\omega})e^{-j2\alpha\omega}.$$

Multiplying both sides by $e^{j\omega\alpha}$,

$$e^{j\omega\alpha}H(e^{j\omega}) = -H^*(e^{j\omega})e^{-j\omega\alpha} = -\overline{e^{j\omega\alpha}H(e^{j\omega})},$$

which implies that both sides must be purely imaginary, so

$$e^{j\omega\alpha}H(e^{j\omega}) = jI(\omega)$$

where $I(\omega)$ is real. Thus, $H(e^{j\omega})$ has the form

$$H(e^{j\omega}) = I(\omega)e^{j\pi/2}e^{-j\omega\alpha}.$$

Therefore, we see that the phase of $H(e^{j\omega})$ consists of a constant term added to a linear function, and the group delay of the system is α , a constant.

Systems with linear phase in this sense are called generalized linear phase.