MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Electrical Engineering and Computer Science

6.341 DISCRETE-TIME SIGNAL PROCESSING
Fall 2005

MIDTERM EXAM

Tuesday, November 8, 2005

SOLUTIONS

Disclaimer: These are not meant to be full solutions, but rather an indication of how to approach each problem.

Problem	Grade	Points	Grader
1		/15	
2 (a)		/12	
2 (b)		/6	
2 (c)		/12	
3 (a)		/10	
3 (b)		/10	
4		/15	
5 (a)		/10	
5 (b)		/10	
Total		/100	

Problem 1 (15%)

A stable system with system function $H(z)$ has the pole-zero diagram shown in Figure 1-1. It can be represented as the cascade of a stable minimum-phase system $H_{min}(z)$ and a stable all-pass system $H_{ap}(z)$.

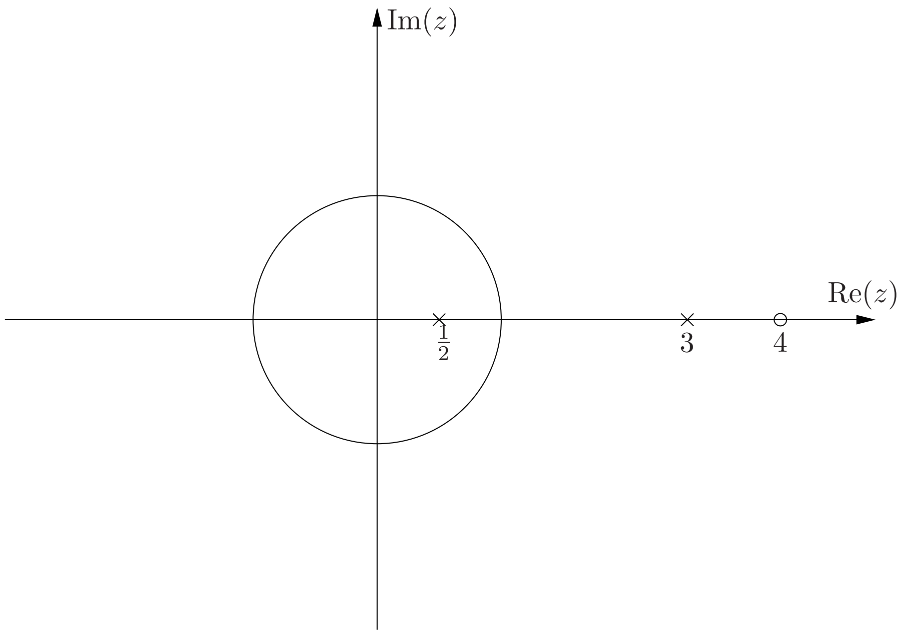


Figure 1-1: Pole-zero diagram for $H(z)$.

Determine a choice for $H_{min}(z)$ and $H_{ap}(z)$ (up to a scale factor) and draw their corresponding pole-zero plots. Indicate whether your decomposition is unique up to a scale factor.

We find a minimum-phase system $H_{min}(z)$ that has the same frequency response magnitude as $H(z)$ up to a scale factor. Poles and zeros that were outside the unit circle are moved to their conjugate reciprocal locations (3 to $\frac{1}{3}$, 4 to $\frac{1}{4}$, ∞ to 0).

$$H_{min}(z) = K_1 \frac{(1 - \frac{1}{4}z^{-1})}{(1 - \frac{1}{2}z^{-1})(1 - \frac{1}{3}z^{-1})}$$

There is no need to include an explicit z term to account for the zero at the origin.

We now include all-pass terms in $H_{ap}(z)$ to move poles and zeros back to their original locations in $H(z)$. The term $\frac{z^{-1}-3}{1-3z^{-1}}$ moves the pole at $\frac{1}{3}$ back to 3, the term z^{-1} moves the zero from 0 to ∞, and so on:

$$H_{ap}(z) = z^{-1} \left(\frac{z^{-1} - 3}{1 - 3z^{-1}} \right) \left(\frac{z^{-1} - \frac{1}{4}}{1 - \frac{1}{4}z^{-1}} \right)$$

The decomposition is unique up to a scale factor. We cannot include additional all-pass terms in $H_{ap}(z)$, since it is not possible for $H_{min}(z)$ to cancel the resulting poles and zeros outside the unit circle.

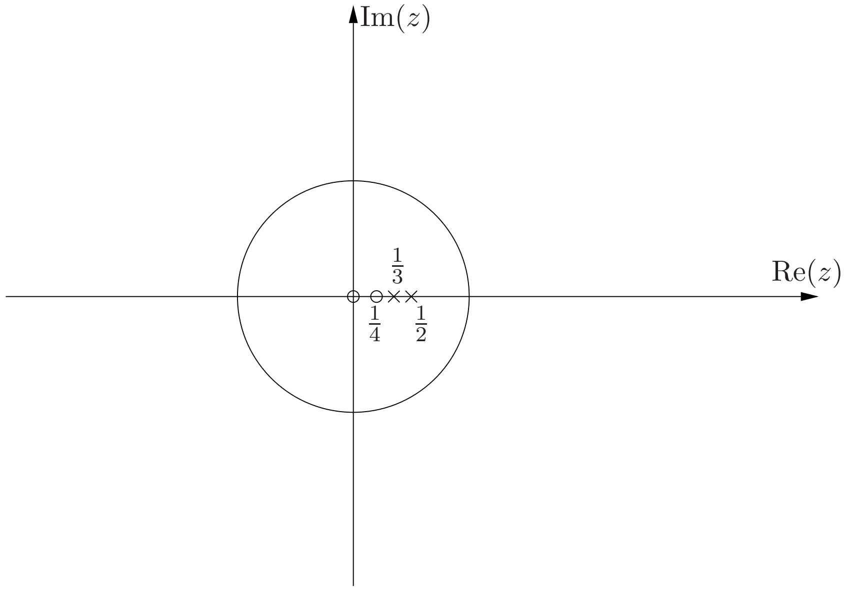


Figure 1-2: Pole-zero diagram for $H_{min}(z)$.

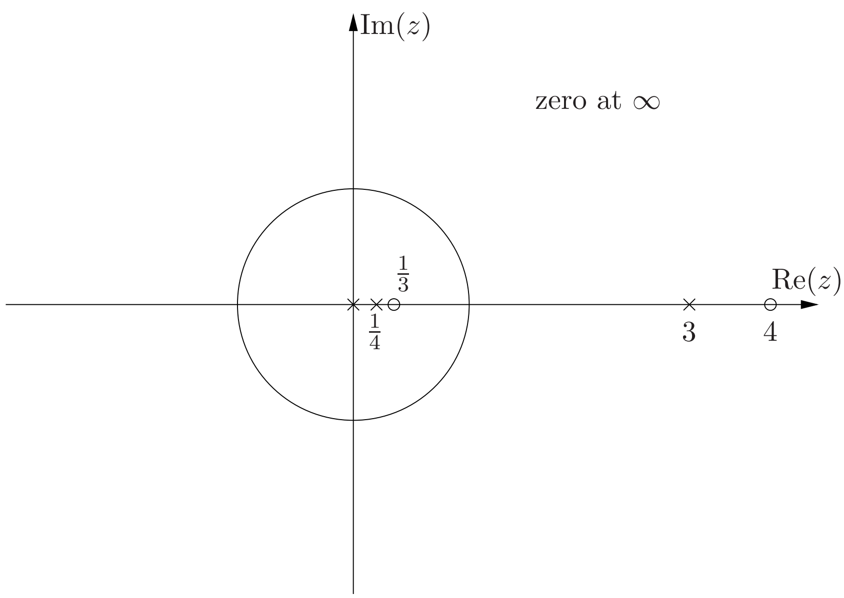


Figure 1-3: Pole-zero diagram for $H_{ap}(z)$.

Problem 2 (30%)

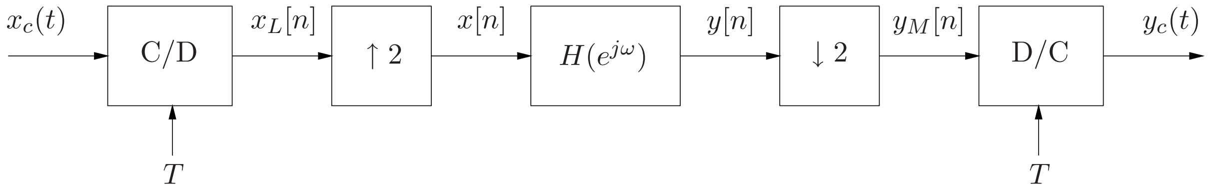


Figure 2-1:

For parts (a) and (b) only, $X_c(j\Omega) = 0$ for $|\Omega| > 2\pi \times 10^3$ and $H(e^{j\omega})$ is as shown in Figure 2-2 (and of course periodically repeats).

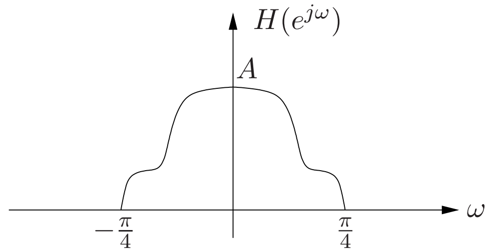


Figure 2-2:

(12%) (a) Determine the most general condition on T, if any, so that the overall continuous-time system from $x_c(t)$ to $y_c(t)$ is LTI.

We can simplify Figure 2-1 by replacing the expander, $H(e^{j\omega})$ and compressor with a single DT LTI system. The impulse response of the equivalent DT LTI system is $h[2n]$, where $h[n]$ is the inverse DTFT of $H(e^{j\omega})$. We saw this result in Problem Set 6 where we derived it by polyphase decomposition and subsequent simplification. The frequency response of the equivalent DT LTI system is $\frac{1}{2}H(e^{j\frac{\omega}{2}}) + \frac{1}{2}H(e^{j(\frac{\omega}{2}-\pi)})$.

Alternatively, we can reach the same conclusion by relating $Y_M(e^{j\omega})$ to $X_L(e^{j\omega})$:

$$Y_M(e^{j\omega}) = \frac{1}{2}\left[H(e^{j\frac{\omega}{2}})X_L(e^{j2\frac{\omega}{2}}) + H(e^{j(\frac{\omega}{2}-\pi)})X_L(e^{j2(\frac{\omega}{2}-\pi)})\right]$$

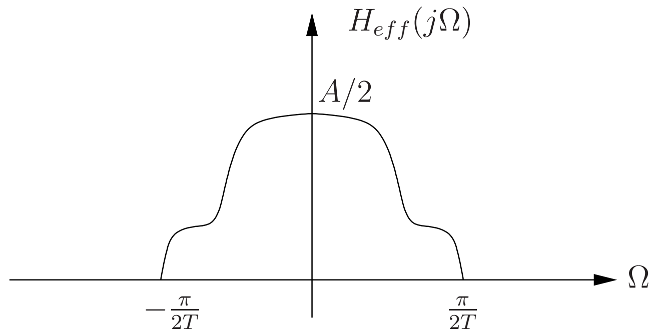
Both $X_L(\cdot)$ terms reduce to $X_L(e^{j\omega})$, the latter because of 2π periodicity, leaving behind the same equivalent frequency response.

The overall CT system will be LTI if there is no aliasing at the C/D converter, or if the DT filter rejects any frequency content contaminated by aliasing. Since the equivalent DT frequency response is 0 for $\frac{\pi}{2} < |\omega| \leq \pi$, the copy of $X_c(j\Omega)$ centred at $\omega = 0$ can extend to $\omega = 2\pi - \frac{\pi}{2} = \frac{3\pi}{2}$ without passing any aliased components. Applying the $\omega = \Omega T$ mapping:

$$\omega_{max} = \Omega_{max}T = (2\pi \times 10^3)T < \frac{3\pi}{2}$$

and so we require that $T < \frac{3}{4} \times 10^{-3}$.

(6%) (b) Sketch and clearly label the overall equivalent continuous-time frequency response $H_{\text{eff}}(j\Omega)$ that results when the condition determined in (a) holds.



(12%) (c) **For this part only** assume that $X_c(j\Omega)$ in Figure 2-1 is bandlimited to avoid aliasing, i.e. $X_c(j\Omega) = 0$ for $|\Omega| \geq \frac{\pi}{T}$. For a general sampling period T, we would like to choose the system $H(e^{j\omega})$ in Figure 2-1 so that the overall continuous-time system from $x_c(t)$ to $y_c(t)$ is LTI for any input $x_c(t)$ bandlimited as above.

Determine the most general conditions on $H(e^{j\omega})$, if any, so that the overall CT system is LTI. Assuming that these conditions hold, also specify in terms of $H(e^{j\omega})$ the overall equivalent continuous-time frequency response $H_{\text{eff}}(j\Omega)$.

The simplification in part (a) yields an equivalent DT LTI system between the C/D and D/C converters. Thus the overall system will be LTI if $x_c(t)$ is appropriately bandlimited to avoid aliasing, as assumed in this part. There are no conditions on $H(e^{j\omega})$.

Problem 3 (20%)

For this problem you may find the information on page 12 useful.

Consider the LTI system represented by the FIR lattice structure in Figure 3-1.

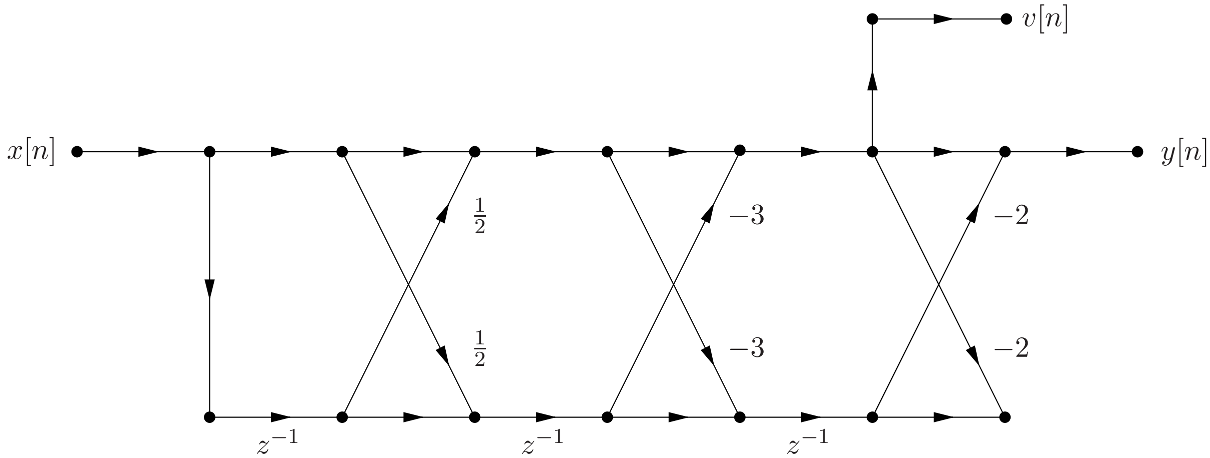


Figure 3-1:

(10%) (a) Determine the system function from the input $x[n]$ to the output $v[n]$ (NOT $y[n]$).

The output $v[n]$ is taken after two stages, so we perform the lattice recursion up to order $p = 2$.

$$k_1 = -\frac{1}{2}, \quad k_2 = 3, \quad k_3 = 2$$

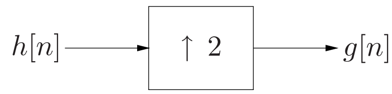
$$a_1^{(1)} = k_1 = -\frac{1}{2}$$

$$\begin{bmatrix} a_1^{(2)} \\ a_2^{(2)} \end{bmatrix} = \begin{bmatrix} a_1^{(1)} \\ 0 \end{bmatrix} - k_2 \begin{bmatrix} a_1^{(1)} \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$$\frac{V(z)}{X(z)} = 1 - z^{-1} - 3z^{-2}$$

Note the change of signs in going from $a_k^{(2)}$ to the system function.

(10%) (b) Let $H(z)$ be the system function from the input $x[n]$ to the output $y[n]$, and let $g[n]$ be the result of expanding the associated impulse response $h[n]$ by 2:

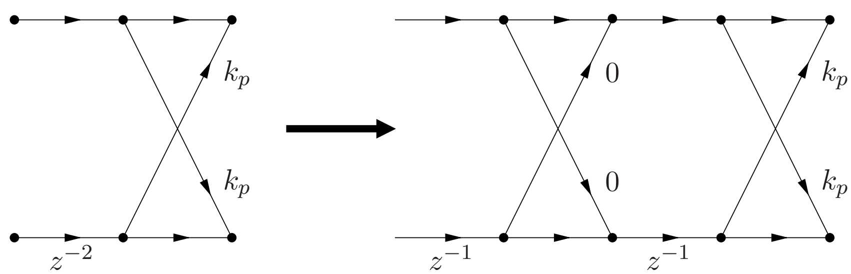


The impulse response $g[n]$ defines a new system with system function $G(z)$.

We would like to implement $G(z)$ using an FIR lattice structure as defined by the figure on page 12. Determine the k-parameters necessary for an FIR lattice implementation of $G(z)$.

Note: You should think carefully before diving into a long calculation.

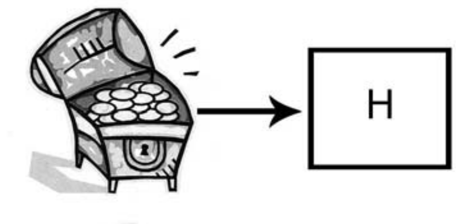
Since $g[n]$ is $h[n]$ expanded by 2, $G(z) = H(z^2)$. We replace z by z^2 in Figure 3-1. We then expand each of the three sections as shown below:



The resulting flowgraph for $G(z)$ is in the form of a 6th-order FIR lattice. We read off the six k-parameters as:

$$\begin{aligned} k_2 &= -\frac{1}{2} \\ k_4 &= 3 \\ k_6 &= 2 \\ k_p &= 0, \quad p = 1, 3, 5 \end{aligned}$$

Problem 4 (15%)



We find in a treasure chest an even-symmetric FIR filter $h[n]$ of length $2L+1$, *i.e.*

$$h[n] = 0 \text{ for } |n| > L,$$

$$h[n] = h[-n].$$

$H(e^{j\omega})$, the DTFT of $h[n]$, is plotted over $-\pi \le \omega \le \pi$ in Figure 4-1.

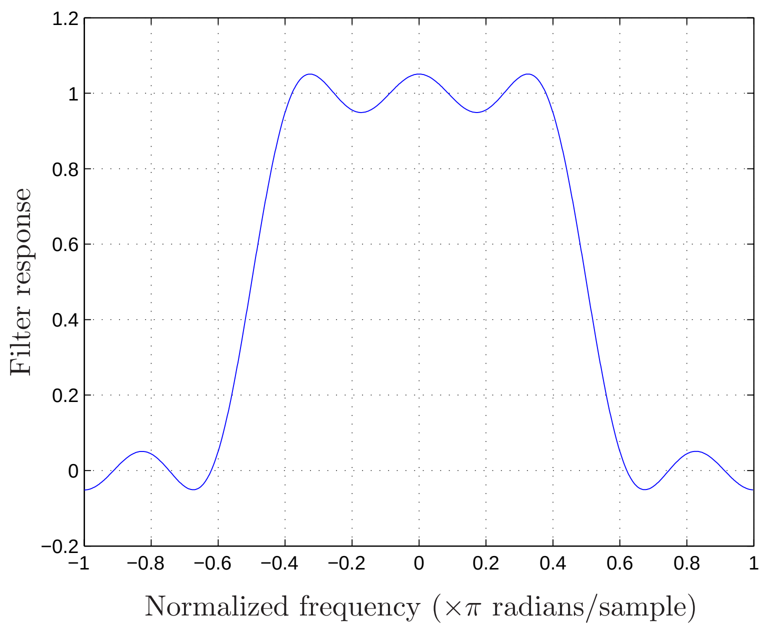


Figure 4-1: Plot of $H(e^{j\omega})$ over $-\pi \le \omega \le \pi$.

What can be inferred from Figure 4-1 about the possible range of values of L? Clearly explain the reason(s) for your answer. Do not make any assumptions about the design procedure that might have been used to obtain $h[n]$.

As discussed in Section 7.4 of OSB, $H(e^{j\omega})$ can be represented as an Lth order polynomial in $\cos\omega$ since $h[n]$ is even-symmetric and finite-length. An Lth order polynomial in $\cos\omega$ can have no more than $L+1$ extrema on $[0,\pi]$. (An Lth order polynomial in x can have no more than $L-1$ extrema in an open interval, but an Lth order polynomial in $\cos\omega$ will always have additional extrema at $\omega = 0$ and $\omega = \pi$ for a total of $L+1$ extrema.)

$H(e^{j\omega})$ has 6 extrema on $[0,\pi]$, so $6 \leq L+1$ or $L \geq 5$.

Problem 5 (20%)

In Figure 5-1, $x[n]$ is a finite sequence of length 1024. The sequence $R[k]$ is obtained by taking the 1024-point DFT of $x[n]$ and compressing the result by 2.

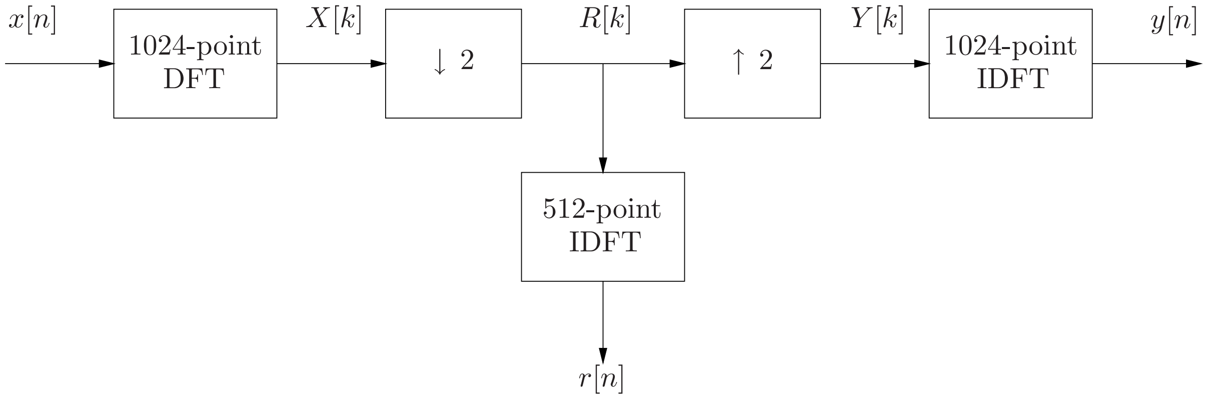


Figure 5-1:

(10%) (a) Choose the most accurate statement for $r[n]$, the 512-point inverse DFT of $R[k]$. Justify your choice in a few concise sentences.

A. $r[n] = x[n],\ 0 \leq n \leq 511$

B. $r[n] = x[2n],\ 0 \leq n \leq 511$

C. $r[n] = x[n] + x[n + 512],\ 0 \leq n \leq 511$

D. $r[n] = x[n] + x[-n + 512],\ 0 \leq n \leq 511$

E. $r[n] = x[n] + x[1023 - n],\ 0 \leq n \leq 511$

In all cases $r[n] = 0$ outside $0 \leq n \leq 511$.

Answer: C

Compressing the 1024-point DFT $X[k]$ by 2 undersamples the DTFT $X(e^{j\omega})$. Undersampling in the frequency domain corresponds to aliasing in the time domain. In this specific case, the second half of $x[n]$ is folded onto the first half, as described by statement C.

(10%) (b) The sequence $Y[k]$ is obtained by expanding $R[k]$ by 2. Choose the most accurate statement for $y[n]$, the 1024-point inverse DFT of $Y[k]$. Justify your choice in a few concise sentences.

A. $y[n] = \begin{cases} \frac{1}{2}\left(x[n] + x[n+512]\right), & 0 \le n \le 511 \\ \frac{1}{2}\left(x[n] + x[n-512]\right), & 512 \le n \le 1023 \end{cases}$

B. $y[n] = \begin{cases} x[n], & 0 \le n \le 511 \\ x[n-512], & 512 \le n \le 1023 \end{cases}$

C. $y[n] = \begin{cases} x[n], & n \text{ even} \\ 0, & n \text{ odd} \end{cases}$

D. $y[n] = \begin{cases} x[2n], & 0 \le n \le 511 \\ x[2(n-512)], & 512 \le n \le 1023 \end{cases}$

E. $y[n] = \frac{1}{2}\left(x[n] + x[1023-n]\right),\ 0 \le n \le 1023$

In all cases $y[n] = 0$ outside $0 \le n \le 1023$.

Answer: A

We can first think about how expanding by 2 in the time domain affects the DFT. Expanding a time sequence $x[n]$ by 2 compresses the DTFT $X(e^{j\omega})$ by 2 in frequency. As a result, the $2N$-point DFT of the expanded sequence samples *two* periods of $X(e^{j\omega})$ and equals two copies of the N-point DFT $X[k]$.

By duality, expanding the DFT $R[k]$ by 2 corresponds to repeating $r[n]$ back-to-back, with an additional scaling by $\frac{1}{2}$. Thus statement A is correct.

Alternatively, $Y[k] = \frac{1}{2}\left(1 + (-1)^k\right) X[k]$. Modulating the DFT by $(-1)^k$ corresponds to a circular time shift of $N/2 = 512$.

END OF EXAM

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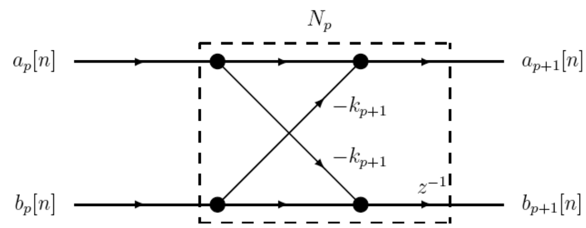

Figure 1: One Section of Lattice Structure

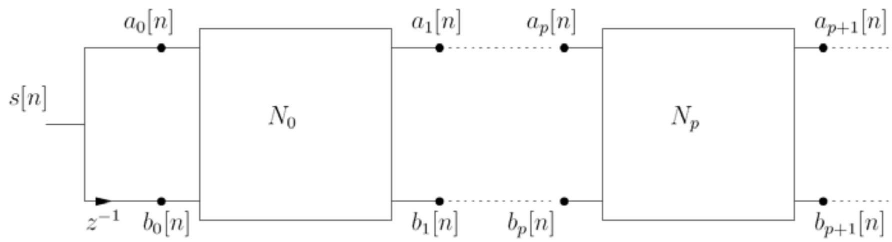

Figure 2:

$$A_{p+1}(z) = A_p(z) - k_{p+1} B_p(z) \tag{1a}$$

$$B_p(z) = z^{-(p+1)} A_p(1/z) \tag{3a}$$

$$A_0(z) = 1 \quad \text{and} \quad B_0(z) = z^{-1} \tag{2}$$

$$a_k^{(p+1)} = \left[a_k^{(p)} - k_{p+1} a_{p-k+1}^{(p)} \right], \qquad k = 1, \ldots, p \tag{8a}$$

$$a_{p+1}^{(p+1)} = k_{p+1} \tag{8b}$$

Define:

$$\underline{\alpha}_p = \left[a_1^{(p)} \ a_2^{(p)} \ \cdots \ a_p^{(p)} \right]^T$$

$$\underline{\check{\alpha}}_p = \left[a_p^{(p)} \ a_{p-1}^{(p)} \ \cdots \ a_1^{(p)} \right]^T$$

Then equations (8) become:

$$\underline{\alpha}_{p+1} = \begin{bmatrix} \underline{\alpha}_p \\ \cdots \\ 0 \end{bmatrix} - k_{p+1} \begin{bmatrix} \underline{\check{\alpha}}_p \\ \cdots \\ -1 \end{bmatrix} \tag{9}$$

Reverse recursion:

$$a_k^{(M-1)} = \frac{a_k^{(M)} + k_M a_{M-k}^{(M)}}{1 - k_M^2}, \qquad k = 1, \ldots, M-1 \tag{11}$$

with $k_M = a_M^{(M)},\ k_{M-1} = a_{M-1}^{(M-1)}$.