

MASSACHUSETTS INSTITUTE OF TECHNOLOGY  
Department of Electrical Engineering and Computer Science

6.341 DISCRETE-TIME SIGNAL PROCESSING  
Fall 2004

**FINAL EXAM**

Monday, December 13, 2004 (DuPont 1:30-4:30pm)

- This is a closed book exam, but **three**  $8\frac{1}{2}'' \times 11''$  **handwritten** sheets of notes (both sides) are allowed.
- Calculators are not allowed.
- Make sure you have all **20** numbered pages of this exam.
- There are **8** problems on the exam.
- The problems are not in order of difficulty. We recommend that you read through all the problems, then do the problems in whatever order suits you best. **Note, too, that usually each of the parts of a problem can be done independently.**
- A correct answer does not guarantee full credit, and a wrong answer does not guarantee loss of credit. You should clearly but concisely indicate your reasoning and **show all relevant work.**
- Please be neat—we can't grade what we can't decipher.
- All answers and work to be graded should be in this exam booklet. Only this answer booklet is to be handed in. **No additional pages will be considered in the grading.** You may want to work things through in the blank areas of the exam first, and then neatly transfer the work you would like us to look at into the exam booklet. Let us know if you need additional scratch paper.
- We will again be using the EGRMU grading strategy. This strategy focuses on your level of understanding of the material associated with each problem. Specifically, when we grade each part of a problem we will do our best to assess, from your work, your level of understanding.
- **Graded Exams and Final Course Grade:**

Graded exams, graded projects part II, and final course grades can be picked up in 36-615 on or after THURSDAY morning, December 16th. If you would like your graded exam and project mailed to you, please leave an addressed, stamped envelope with us at the end of the exam. We'll use the envelope as is, so please be sure to address it properly and with enough postage. We guarantee that we'll put it into the proper mailbox, but we don't guarantee anything beyond that.

OUT OF CONSIDERATION FOR THE 6.341 STAFF, UNDER NO CIRCUMSTANCES WILL GRADES BE AVAILABLE BY PHONE OR EMAIL. PLEASE DON'T EVEN ASK.
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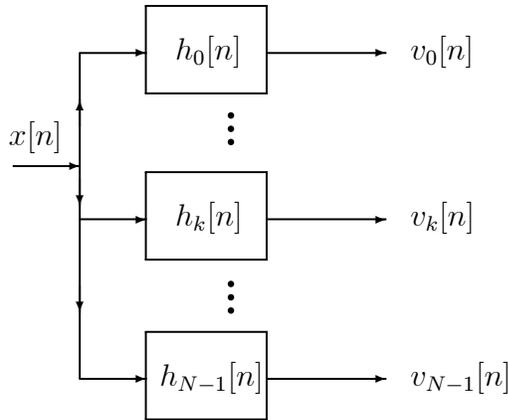
**NAME:** \_\_\_\_\_

Problem	Grade	Points	Grader
1			
2			
3			
4			
5 (a)			
5 (b)			
6			
7			
8 (a)			
8 (b)			
8 (c)			
8 (d)			
Total			

**COURSE GRADE:**

**Problem 1** [ 10% ]

The system in figure 1-1, uses a modulated filter bank for spectral analysis. The lowpass impulse response  $h_0[n]$  is sketched in figure 1-2.



$$h_k[n] = e^{j\omega_k n} h_0[n], \quad \omega_k = \frac{2\pi k}{N}, \quad \text{where } k = 0, 1, \dots, N - 1$$

$$h_0[n] = \text{lowpass prototype filter}, \quad H_k(z) = H_0(e^{-j\frac{2\pi k}{N}} z)$$

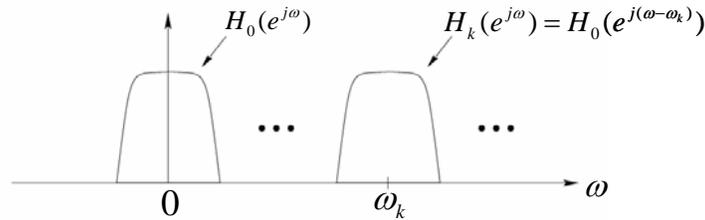


Figure 1-1:

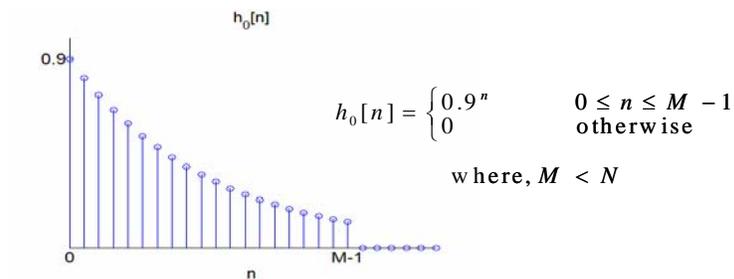


Figure 1-2:

Name: \_\_\_\_\_

An alternative system for spectral analysis is shown in figure 1-3. Determine  $w[n]$  so that  $G[k] = v_k[0]$ , for  $k = 0, 1, \dots, N - 1$ .

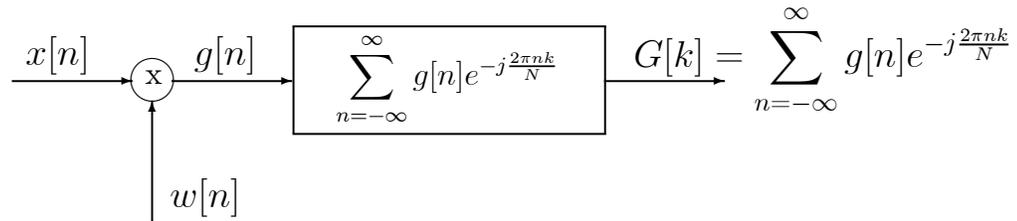


Figure 1-3

**Work to be looked at and answer:**

Note: As with all the problems, a correct answer without explanation and related work will not guarantee full credit.

**Problem 2** [ 8% ]

Consider the stable LTI system with system function

$$H(z) = \frac{1 + 4z^{-2}}{1 - \frac{1}{4}z^{-1} - \frac{3}{8}z^{-2}}$$

The system function  $H(z)$  can be factored such that

$$H(z) = H_{\min}(z)H_{ap}(z)$$

where  $H_{\min}(z)$  is a minimum phase system, and  $H_{ap}(z)$  is an allpass system, i.e.,

$$|H_{ap}(e^{j\omega})| = 1$$

Sketch the pole-zero diagrams for  $H_{\min}(z)$  and  $H_{ap}(z)$ . Be sure to label the positions of all the poles and zeros. Also, indicate the region of convergence for  $H_{\min}(z)$  and  $H_{ap}(z)$ .

**Work to be looked at and answer:**

Name: \_\_\_\_\_

**Problem 3** [ 8% ]

The block diagram in figure 3-1 represents a system that we would like to implement. Determine a block diagram of an equivalent system consisting of a cascade of LTI systems, compressor blocks, and expander blocks which results in the minimum number of multiplications per output sample.

**Note:** By "equivalent system" we mean that it produces the same output sequence for any given input sequence.

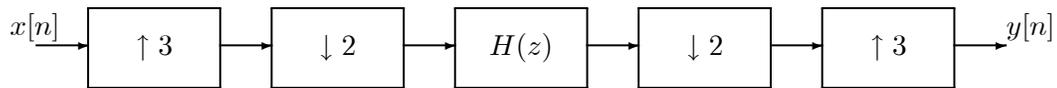


Figure 3-1

$$H(z) = \frac{z^{-6}}{7 + z^{-6} - 2z^{-12}}$$

**Work to be looked at and answer:**

**Problem 4** [ 8% ]

Consider a colored wide sense stationary stochastic signal  $s[n]$  which we desire to whiten using the system in figure 4-1:

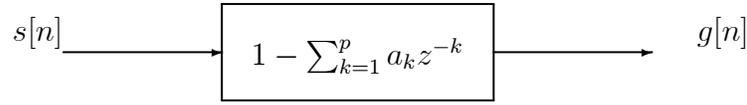


Figure 4-1

In designing the optimal whitening filter for a given order  $p$ , we pick  $a_k^{(p)}$ ,  $k = 1, \dots, p$  that solve the following equations, where  $\phi_s[m]$  is the autocorrelation of  $s[n]$ .

$$\begin{bmatrix} \phi_s[0] & \phi_s[1] & \dots & \phi_s[p-1] \\ \phi_s[1] & \phi_s[0] & \dots & \\ \vdots & & \ddots & \\ \phi_s[p-1] & \dots & & \phi_s[0] \end{bmatrix} \begin{bmatrix} a_1^{(p)} \\ a_2^{(p)} \\ \vdots \\ a_p^{(p)} \end{bmatrix} = \begin{bmatrix} \phi_s[1] \\ \phi_s[2] \\ \vdots \\ \phi_s[p] \end{bmatrix}$$

It is known that the optimal 2<sup>nd</sup> order whitening filter for  $s[n]$  is  $H_2(z) = 1 + \frac{1}{4}z^{-1} - \frac{1}{8}z^{-2}$ , (i.e.  $a_1^{(2)} = -\frac{1}{4}$ ,  $a_2^{(2)} = \frac{1}{8}$ ), which we implement in the 2<sup>nd</sup> order lattice structure in figure 4-2:

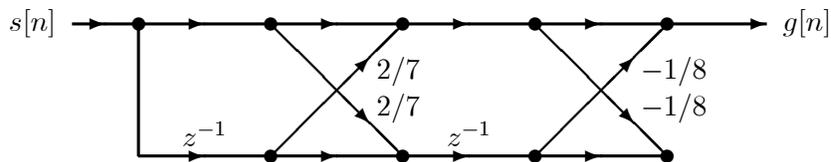


Figure 4-2: Lattice Structure for 2<sup>nd</sup> Order System

We decided that a second order implementation is not sufficient for our application, and we would like to use a 4<sup>th</sup> order system, with transfer function

$$H_4(z) = 1 - \sum_{k=1}^4 a_k^{(4)} z^{-k}$$

We implement this system with the lattice structure in figure 4-3:

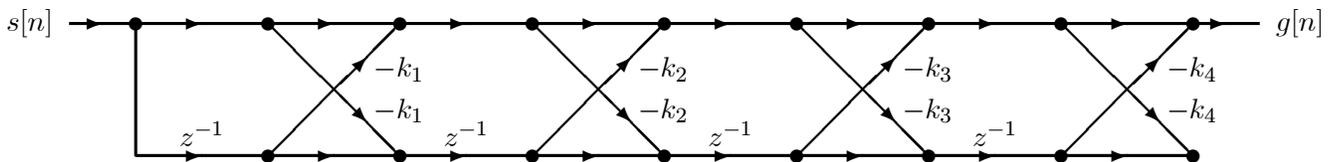


Figure 4-3: Lattice Structure for 4<sup>th</sup> Order System

**Name:** \_\_\_\_\_

Determine which, if any of  $H_4(z)$ ,  $k_1$ ,  $k_2$ ,  $k_3$ ,  $k_4$  can be exactly determined from the information given above. Explain why you cannot determine the remaining, if any, parameters.

**Note:** For this problem you may find useful the lecture slide which we have reproduced on page 20.

**Work to be looked at and answer:**

**Problem 5** [ 16% ]

Each part of this problem may be solved independently. All parts use the signal  $x[n]$  shown in figure 5-1.

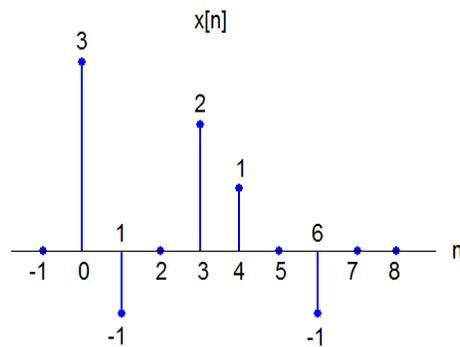


Figure 5-1:

[%8] (a) Let  $X(e^{j\omega})$  be the DTFT of  $x[n]$ . Define

$$R[k] = X(e^{j\omega})|_{\omega=\frac{2\pi k}{4}}, \quad 0 \leq k \leq 3$$

Sketch the signal  $r[n]$  which is the four-point inverse DFT of  $R[k]$ .

**Work to be looked at and answer:** (Not necessary to derive the result mathematically, but explain your result.)

Name: \_\_\_\_\_

[%8] (b) Let  $X[k]$  be the eight-point DFT of  $x[n]$ , and let  $H[k]$  be the eight-point DFT of the impulse response  $h[n]$  shown in figure 5-2. Define  $Y[k] = X[k]H[k]$  for  $0 \leq k \leq 7$ . Sketch  $y[n]$ , the eight-point DFT of  $Y[k]$ .

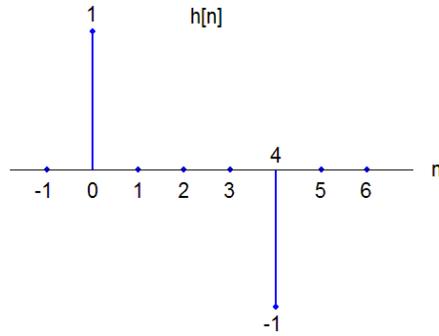


Figure 5-2:

**Work to be looked at and answer:**

**Problem 6** [ 10% ]

Consider a time-limited continuous-time signal  $x_c(t)$  whose duration is  $100ms$ . Assume that this signal has a bandlimited Fourier transform such that  $X_c(j\Omega) = 0$  for  $|\Omega| \geq 2\pi(10,000)\text{rad/s}$ ; i.e., assume that aliasing is negligible. We want to compute samples of  $X_c(j\Omega)$  with  $5Hz$  spacing over the interval  $0 \leq \Omega \leq 2\pi(10,000)$ . This can be done with a 4000-point DFT. Specifically, we want to obtain a 4000-point sequence  $x[n]$  for which the 4000-point DFT is related to  $X_c(j\Omega)$  by:

$$X[k] = \alpha X_c(j2\pi \cdot 5 \cdot k), \quad k = 0, 1, \dots, 1999, \quad (1)$$

where  $\alpha$  is a known scale factor. The following method is proposed to obtain a 4000-point sequence whose DFT gives the desired samples of  $X_c(j\Omega)$ .  $x_c(t)$  is sampled with a sampling period of  $T = 50\mu s$ . The resulting 2000-point sequence is used to form the sequence  $\hat{x}[n]$  as follows:

$$\hat{x}[n] = \begin{cases} x_c(nT), & 0 \leq n \leq 1999, \\ x_c((n - 2000)T), & 2000 \leq n \leq 3999, \\ 0, & \text{otherwise.} \end{cases} \quad (2)$$

The 4000-point DFT  $\hat{X}[k]$  of this sequence is computed. For this method, determine how  $\hat{X}[k]$  is related to  $X_c(j\Omega)$ . Indicate this relationship in a sketch for a “typical” Fourier transform  $X_c(j\Omega)$ . Explicitly state whether or not  $\hat{X}[k]$  is the desired result, i.e. whether  $\hat{X}[k]$  equals  $X[k]$  as specified in eqn(1).

**Work to be looked at and answer:**

Note: As with all the problems, a correct answer without explanation and related work will not guarantee full credit.

Name: \_\_\_\_\_

**Work to be looked at and answer for problem 6:**

Note: This space may or may not be needed, but is not to be used for any other problem.

**Problem 7** [ 10% ]

The system in figure 7-1 computes an  $N$ -point (where  $N$  is an even number) DFT  $X[k]$  of an  $N$ -point sequence  $x[n]$  by decomposing  $x[n]$  into two  $N/2$ -point sequences  $g_1[n]$  and  $g_2[n]$ , computing the  $N/2$ -point DFT's  $G_1[k]$  and  $G_2[k]$ , and then combining these to form  $X[k]$ .

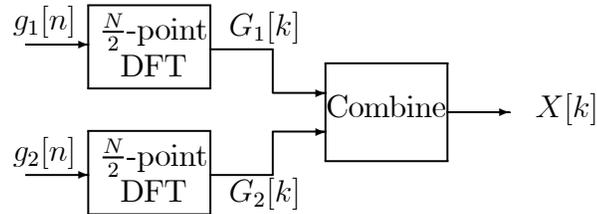


Figure 7-1

If  $g_1[n]$  is the even-indexed values of  $x[n]$  and  $g_2[n]$  is the odd-indexed values of  $x[n]$  i.e.  $g_1[n] = x[2n]$  and  $g_2[n] = x[2n + 1]$  then  $X[k]$  will be the DFT of  $x[n]$ .

In using the system in figure 7-1 an error is made in forming  $g_1[n]$  and  $g_2[n]$ , such that  $g_1[n]$  is **incorrectly** chosen as the odd-indexed values and  $g_2[n]$  as the even indexed values but  $G_1[k]$  and  $G_2[k]$  are still combined as in figure 7-1 and the incorrect sequence  $\hat{X}[k]$  results. Express  $\hat{X}[k]$  in terms of  $X[k]$

**Work to be looked at and answer:**

Name: \_\_\_\_\_

**Problem 8** [ 30% ] **Note:** each part of this problem is independent of the others.

The current CD technology, from production to playback, can be approximated with the block diagram in figure 8-1.

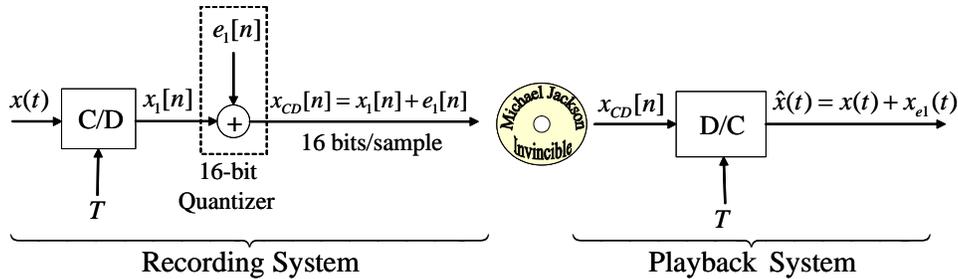


Figure 8-1:

A C/D is defined through the following relationships between the input  $x_c(t)$  and the output  $x_d[n]$ :

$$x_d[n] = x_c(nT) \text{ and } R_{x_d x_d}[m] = R_{x_c x_c}(mT)$$

A D/C is defined through the following relationships between the input  $x_d[n]$  and the output  $x_c(t)$ :

$$x_c(t) = \sum_{n=-\infty}^{+\infty} x_d[n] \frac{\sin[\pi(t - nT)/T]}{\pi(t - nT)/T} \text{ and } R_{x_c x_c}(\tau) = \sum_{n=-\infty}^{+\infty} R_{x_d x_d}[m] \frac{\sin[\pi(\tau - mT)/T]}{\pi(\tau - mT)/T}$$

$x_c(t)$  is a signal bandlimited to  $\pm\pi/T$ , i.e.  $X(j\Omega) = 0$  for  $|\Omega| \geq \pi/T$ .

Assume the additive noise model for a quantizer holds, i.e.  $e_1[n]$  in figure 8-1 is zero-mean, white, has variance  $\sigma_{e_1}^2$  and is uncorrelated with  $x_1[n]$ .

[6%] (a) Determine  $E\{x_{e_1}^2(t)\}$ , the power of the quantization noise at the output of the playback system.

**Work to be looked at and answer:**

Sony and Philips offer a new CD format called Superaudio CD. The new format can be approximately described with the block diagram in figure 8-2.

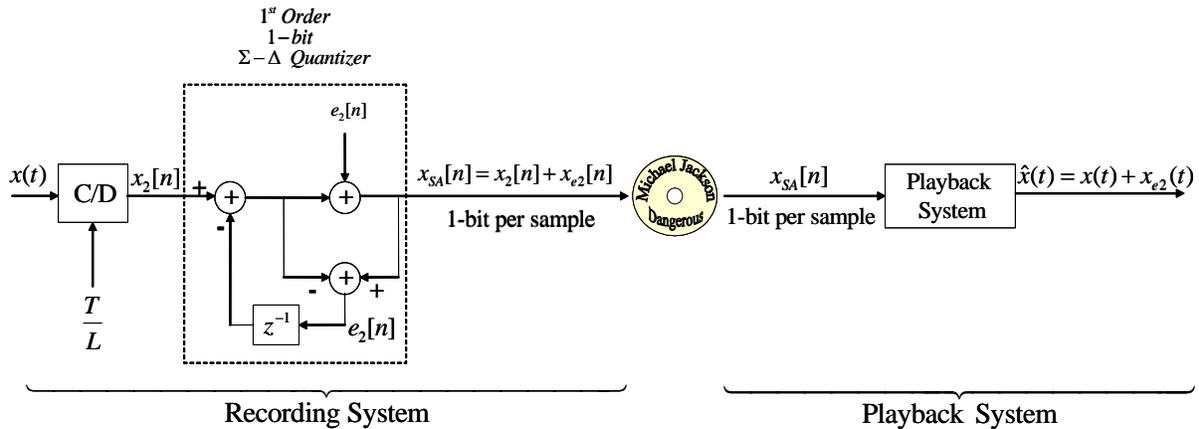


Figure 8-2:

where  $e_2[n]$  in figure 8-2 is zero-mean, white, has variance  $\sigma_{e_2}^2$  and is uncorrelated with  $x_1[n]$ . The power spectral density of  $x(t)$ ,  $P_x(j\Omega)$ , is as shown in figure 8-3.

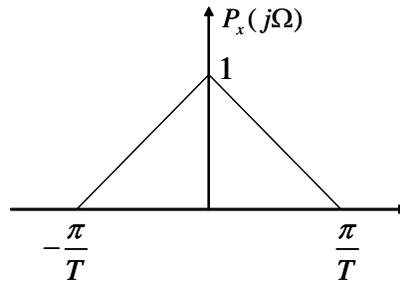


Figure 8-3:

- [8%] (b) In the absence of quantization (i.e. if  $e_2[n] = 0$ ), the playback system in figure 8-2 should reconstruct  $x(t)$  exactly. In the presence of quantization, the overall system should minimize  $E\{x_{e_2}^2(t)\}$ , the quantization noise power at the output. Using the components shown in figure 8-4, design the minimum cost playback system to reconstruct  $x(t)$  from  $x_{SA}[n]$  that satisfies these requirements. Make sure you specify the values for all the parameters of all the components you include in your design.

Note: In the components table all the components have inputs and outputs of arbitrary precision, and can be used with any input. The only exception is with the component named "1-bit D/C converter" that can only be used with a bitstream input, i.e. a signal which has already been quantized to 1-bit per sample. This component cannot be used with any other kind of input.

Name: \_\_\_\_\_

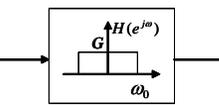
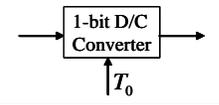
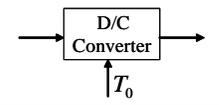
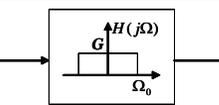
	Component	Parameters	Cost in \$
	Expander	Ratio: R	1
	Compressor	Ratio: R	1
	Discrete-Time Ideal Lowpass Filter	Gain: G Cutoff Frequency: $\omega_0$	1
	1-bit D/C Converter	Rate: $T_0$	1
	D/C Converter	Rate: $T_0$	15
	Continuous-Time Ideal Lowpass Filter	Gain: G Cutoff Frequency: $\Omega_0$	20

Figure 8-4:

Work to be looked at and answer for part (b):

[8%] (c) Repeat part (b) assuming the cost of an analog lowpass filter is now \$1. Figure 8-4 is repeated here with the new cost for the analog filter.

Again note that: In the components table all the components have inputs and outputs of arbitrary precision, and can be used with any input. The only exception is with the component named "1-bit D/C converter" that can only be used with a bitstream input, i.e. a signal which has already been quantized to 1-bit per sample. This component cannot be used with any other kind of input.

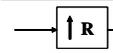
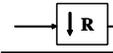
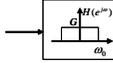
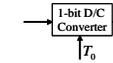
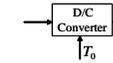
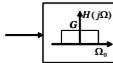
	Component	Parameters	Cost in \$
	Expander	Ratio: R	1
	Compressor	Ratio: R	1
	Discrete-Time Ideal Lowpass Filter	Gain: G Cutoff Frequency: $\omega_b$	1
	1-bit D/C Converter	Rate: $T_0$	1
	D/C Converter	Rate: $T_0$	15
	Continuous-Time Ideal Lowpass Filter	Gain: G Cutoff Frequency: $\Omega_b$	1

Figure 8-5:

Work to be looked at and answer:

Name: \_\_\_\_\_

[8%] (d) With present technology, reconstructing the signal from traditional CDs (i.e. the reconstruction system in figure 8-1) consumes 2 Watts, while the Superaudio CD (i.e. the reconstruction system in figure 8-2) consumes 1 Watts (1 Watt = 1 Joule/sec). However, the energy consumption of reading 1000 bits from the disc is the same.

(i) If  $T = 1/40kHz$  and  $L = 64$ , determine how many bits per second each format requires to store  $x(t)$ .

(ii) If  $T = 1/40kHz$  and  $L = 64$ , as above, determine the energy consumption in Joules per 1000 bits of storage, below which the Superaudio CD format is more power efficient than the traditional CD format in playing back the signal.

**END OF EXAM**

YOU CAN USE THIS PAGE AS SCRATCH PAPER BUT NOTHING ON THIS PAGE WILL BE CONSIDERED DURING GRADING

### Levinson-Durbin Recursion

$$\sum_{k=1}^p a_k \phi_s[i-k] = \phi_s[i] \quad i = 1, 2, \dots, p$$

$$T_p = \begin{bmatrix} \phi_s[0] & \phi_s[1] & \dots & \phi_s[p-1] \\ \phi_s[1] & \phi_s[0] & \dots & \\ \vdots & & & \\ \phi_s[p-1] & \dots & & \phi_s[0] \end{bmatrix}$$

$$\alpha_p = [a_1, a_2 \dots a_p]^T \quad r_p = [\phi_s[1] \phi_s[2] \dots \phi_s[p]]^T$$

$$\beta_p = [a_p, a_{p-1} \dots a_1]^T \quad \rho_p = [\phi_s[p] \phi_s[p-1] \dots \phi_s[1]]^T$$

$$k_{p+1} = \frac{\phi_s[p+1] - (\rho_p)^T \alpha_p}{\phi_s[0] - (r_p)^T \alpha_p}$$

$$\varepsilon_p = -k_{p+1} \beta_p$$

$$\alpha_{p+1} = \begin{bmatrix} \alpha_p \\ 0 \end{bmatrix} + \begin{bmatrix} \varepsilon_p \\ k_{p+1} \end{bmatrix} = \begin{bmatrix} a_1^{(p)} \\ a_2^{(p)} \\ \vdots \\ a_p^{(p)} \\ 0 \end{bmatrix} - k_{p+1} \begin{bmatrix} a_p^{(p)} \\ a_{p-1}^{(p)} \\ \vdots \\ a_1^{(p)} \\ -1 \end{bmatrix} \quad (7)$$

$a_k^{(p)}$  is the  $a_k$ th coefficient for the  $p$ th order filter.