Massachusetts Institute of Technology
Department of Electrical Engineering and Computer Science

6.341: Discrete-Time Signal Processing

Fall 2005

Problem Set 9 Solutions

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Problem 9.1 (OSB 10.12)

Answer: $N \geq 1600$ (as in the back of the text).

$X_c(j\Omega)$ is sufficiently bandlimited to avoid aliasing, so $X(e^{j\omega})$ corresponds to $X(j\Omega)$ rescaled with $\Omega = \frac{\omega}{T}$. The N-point DFT $X[k]$ computes samples of $X(e^{j\omega})$ at frequencies evenly spaced by $\frac{2\pi}{N}$. The equivalent spacing with respect to continuous-time frequency Ω is thus $\frac{2\pi}{NT}$. With a sampling rate of $1/T = 8$ kHz, we require:

$$\begin{aligned} \frac{2\pi}{NT} &\leq 2\pi \cdot 5 \\ N &\geq 1600 \end{aligned}$$

Since the minimum N of 1600 is greater than 1000, the length of $x[n]$, we need to zero-pad $x[n]$ before computing the DFT.

Problem 9.2 (OSB 10.14)

Answer: $x_2[n]$, $x_3[n]$, and $x_6[n]$ could be $x[n]$ (as in the back of the text).

The two non-zero DFT coefficients at $k = 8$ and $k = 16$ correspond to the following frequencies:

$$\begin{aligned} \omega_1 &= \frac{(2\pi)(8)}{128} = \frac{\pi}{8} \\ \omega_1 &= \frac{(2\pi)(16)}{128} = \frac{\pi}{4} \end{aligned}$$

$x_1[n]$ and $x_4[n]$ are eliminated because their frequencies are inconsistent with the figure.

The magnitude of $V[16]$ is about 3 times that of $V[8]$, which eliminates $x_5[n]$ where the ratio of amplitudes is reversed.

$x_2[n]$, $x_3[n]$, and $x_6[n]$ are related by phase shifts in the second term, so the amplitudes of their spectral peaks are equal in magnitude and differ in phase. But since the figure only shows the magnitude of $V[k]$, they cannot be distinguished from one another and are all consistent.

Problem 9.3 (OSB 10.24)

(a) We relate the DFT $X[k]$ of the discrete-time signal $x[n]$ to the continous-time Fourier transform $X_c(j\Omega)$ of the continuous-time signal $x_c(t)$. Since $x[n]$ is obtained by sampling $x_c(t)$ and $x_c(t)$ is appropriately bandlimited,

$$X\left(e^{j\omega}\right) = \frac{1}{T} X_c\left(j\frac{\omega}{T}\right) \qquad \text{for } -\pi \leq \omega \leq \pi$$

which is equivalent to

$$X\left(e^{j\omega}\right) = \begin{cases} \frac{1}{T} X_c\left(j\frac{\omega}{T}\right), & 0 \leq \omega < \pi \\ \frac{1}{T} X_c\left(j\frac{\omega - 2\pi}{T}\right), & \pi \leq \omega < 2\pi \end{cases}$$

Since the DFT is a sampled version of $X\left(e^{j\omega}\right)$,

$$X[k] = X\left(e^{j\omega}\right)|_{\omega = \frac{2\pi k}{N}}, \qquad 0 \leq k \leq N-1$$

we find

$$X[k] = \begin{cases} \frac{1}{T} X_c\left(j\frac{2\pi k}{NT}\right), & 0 \leq k < \frac{N}{2} \\ \frac{1}{T} X_c\left(j\frac{2\pi(k-N)}{NT}\right), & \frac{N}{2} \leq k < N-1 \end{cases}$$

The second case above is necessary to relate the second half of $X[k]$ to the negative frequencies in $X_c(j\Omega)$.

The effective frequency spacing is

$$\Delta\Omega = \frac{2\pi}{NT} = \frac{2\pi}{(1000)(1/20,000)} = 2\pi(20)\text{rad/s}$$

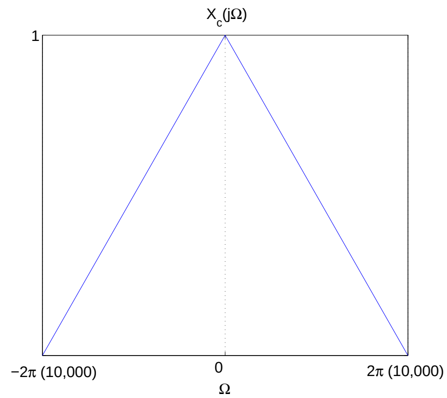
(b) Next, we determine if the designer's assertion that

$$Y[k] = \alpha X_c(j2\pi \cdot 10 \cdot k), \qquad k = 0, 1, \ldots, 500$$

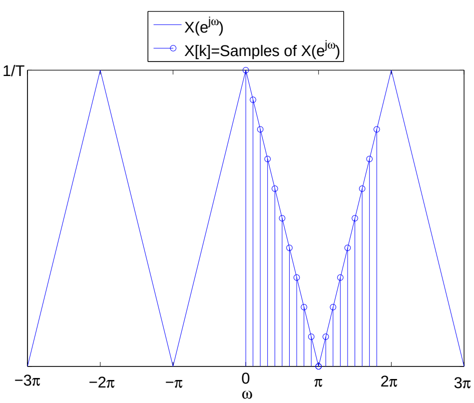
is correct.

In words, we wish to "zoom in" on the lower half of $X_c(j\Omega)$, i.e. $|\Omega| \leq 2\pi(5000)$. We then sample just the lower half of the spectrum at 1000 points to obtain a finer view than what we had with $X[k]$. The designer reasons that the "zooming" or frequency expansion can be accomplished by downsampling the sampled sequence $x[n]$ by 2. Downsampling involves low-pass filtering to remove the upper half of the spectrum, followed by compression. However, the designer incorrectly assumes that ideal low-pass filtering can be achieved by multiplication with an "ideal" response in the DFT domain. We will see the consequences of this incorrect assumption.

Assume that the spectrum of the original signal $x_c(t)$ looks like:



We obtain $x[n]$ by sampling $x_c(t)$. $X(e^{j\omega})$ and $X[k]$ are shown below:

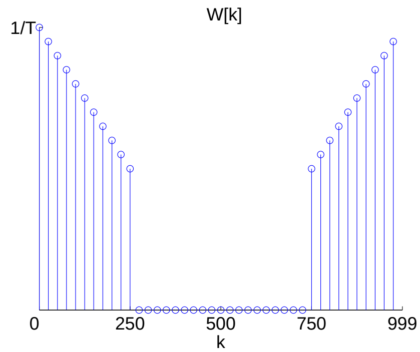


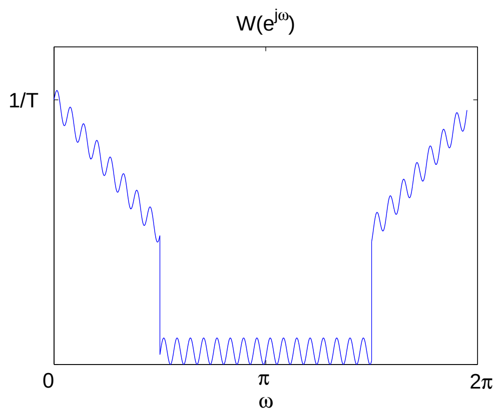
We low-pass filter $X[k]$ in the DFT domain to form $W[k]$:

$$W[k] = \begin{cases} X[k], & 0 \le k \le 250 \\ 0, & 251 \le k \le 749 \\ X[k], & 750 \le k \le 999 \end{cases}$$

and we find $w[n]$ as the inverse DFT of $W[k]$.

However, the DTFT $W(e^{j\omega})$ of $w[n]$ will not correspond to passing $x[n]$ through an ideal low-pass filter. $W(e^{j\omega})$ does pass through the points sampled by $W[k]$, so it is equal to $X(e^{j\omega_k})$ at the points $0 \le k \le 250$ and $750 \le k \le 999$, and is zero for $251 \le k \le 749$. It does not equal $X(e^{j\omega})$ in between the sampled points in the passband, and in particular it is not zero in between the sampled points in the stopband. $W(e^{j\omega})$ is pictured below:





We obtain $y[n]$ by compressing $w[n]$ by 2 and padding the result with 500 zeroes. The DTFT $Y(e^{j\omega})$ is given by

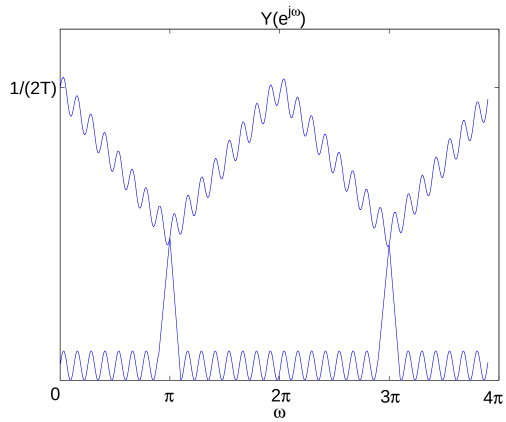
$$Y(e^{j\omega}) = \frac{1}{2}W(e^{j\frac{\omega}{2}}) + \frac{1}{2}W(e^{j\frac{\omega-2\pi}{2}})$$

and is depicted below.

$Y[k]$ is equal to samples of $Y(e^{j\omega})$:

$$Y[k] = Y(e^{j\omega})|_{\omega=\frac{2\pi k}{N}} = \frac{1}{2}W(e^{j\frac{2\pi}{N}\frac{k}{2}}) + \frac{1}{2}W(e^{j\frac{2\pi}{N}\frac{(k-N)}{2}})$$

We restrict our attention to $k = 0, 1, \ldots, 500$. For even values of k, $Y[k]$ only involves points sampled by $W[k]$, which are either proportional to $X_c(j\Omega)$ through $X(e^{j\omega})$ or are zero. However, for odd values of k, $Y[k]$ involves values of $W(e^{j\omega})$ in between the points sampled by $W[k]$, including aliasing from non-zero values of $W(e^{j\omega})$ in the stopband. In addition, $Y[500]$ also contains aliasing because it is the sum of $W[250]$ and $W[-250] =$



$W[750] \neq 0$.

$$Y[k] = \begin{cases} \frac{1}{2T} X_c(j2\pi \cdot 10 \cdot k), & k \text{ even}, k \neq 500 \\ \frac{1}{2T} W(e^{j\frac{\pi k}{1000}}) + \frac{1}{2T} W(e^{j\frac{\pi k}{1000} - \pi}), & k \text{ odd} \\ \frac{1}{2T} X_c(j2\pi \cdot 10 \cdot k) + \frac{1}{2T} X_c(-j2\pi \cdot 10 \cdot k), & k = 500 \end{cases}$$

In other words, the even-indexed DFT samples are not aliased, but the odd indexed samples (and $k = 500$) are aliased. The designer's assertion is not correct.

Problem 9.4 (OSB 10.31)

(a) Sampling the continuous-time input signal $x(t) = e^{j(3\pi/8)10^4 t}$ with a sampling period $T = 10^{-4}$ yields the discrete-time signal:

$$x[n] = x(nT) = e^{j\frac{3\pi n}{8}}$$

In order for $X_w[k]$ to be nonzero at exactly one value of k, the frequency of the complex exponential in $x[n]$ must correspond exactly to the frequency of a DFT bin $\omega_k = \frac{2\pi k}{N}$ for some k.

$$\begin{aligned} \frac{3\pi}{8} &= \frac{2\pi k}{N} \\ N &= \frac{16k}{3} \end{aligned}$$

The smallest value of k for which N is an integer is $k = 3$. Thus, the smallest value of N such that $X_w[k]$ is nonzero at exactly one value of k is

$$N = 16$$

(b) The rectangular windows, $w_1[n]$ and $w_2[n]$, differ only in their lengths, which are 32 and 8 respectively. Recall that the Fourier transform of a shorter window has a wider mainlobe and higher sidelobes compared to that of a longer window. Since the DFT is a sampled version of the DTFT, we try to use these features to distinguish the two plots. We notice that the second plot, Figure P10.31-3, appears to have a wider mainlobe and higher sidelobes. As a result, we conclude that Figure P10.31-2 corresponds to $w_1[n]$, and Figure P10.31-3 corresponds to $w_2[n]$.

(c) A simple technique to estimate the value of ω_0 is to find the value of k where $|X_w[k]|$ is largest. Call this index $\hat{k}_0$. The estimate is then:

$$\widehat{\omega}_0 = \frac{2\pi\hat{k}_0}{N}$$

The corresponding value of $\widehat{\Omega}_0$ is

$$\widehat{\Omega}_0 = \frac{2\pi\hat{k}_0}{NT}$$

This estimate is not exact, since the peak of the Fourier transform magnitude $|X_w(e^{j\omega})|$ could occur between two DFT samples. The maximum possible error $\Delta\Omega_{\max}$ in the estimate is one half of the frequency resolution of the DFT.

$$\Delta\Omega_{\max} = \frac{1}{2}\frac{2\pi}{NT} = \frac{\pi}{NT}$$

From Figure P10.31-2, $k = 6$, and with the system parameters $N = 32$ and $T = 10^{-4}$,

$$\widehat{\Omega}_0 \pm \Delta\Omega_{\max} = 11781 \pm 982 \text{ rad/s} = 1875 \pm 156 \text{ Hz}$$

(d) The following procedure provides a precise estimate of Ω_0, starting from the coarse estimate in part (c). Other procedures are also possible.

We seek an algebraic expression for the N-point DFT $X_w[k]$. We first find the Fourier transform of $x_w[n] = x[n]w[n]$, where $w[n]$ is an M-point rectangular window and M is not necessarily equal to N. Since $x[n]$ is a pure complex exponential with frequency ω_0, $X_w(e^{j\omega})$ is equal to the Fourier transform of an M-point rectangular window shifted in frequency by ω_0:

$$X_w(e^{j\omega}) = \frac{\sin\left(\frac{(\omega-\omega_0)M}{2}\right)}{\sin\left(\frac{(\omega-\omega_0)}{2}\right)} e^{-j\frac{(\omega-\omega_0)(M-1)}{2}}$$

Note that $X_w(e^{j\omega})$ has generalized linear phase. We find $X_w[k]$ by evaluating the above expression at frequencies $\omega = \frac{2\pi k}{N}$ for $k = 0, 1, \ldots, N-1$:

$$X_w[k] = \frac{\sin\left(\frac{\left(\frac{2\pi k}{N}-\omega_0\right)M}{2}\right)}{\sin\left(\frac{\left(\frac{2\pi k}{N}-\omega_0\right)}{2}\right)} e^{-j\frac{\left(\frac{2\pi k}{N}-\omega_0\right)(M-1)}{2}}$$

We know the wrapped phase of $X_w[k]$, given by:

$$\measuredangle X_w[k] = \left(\omega_0 - \frac{2\pi k}{N}\right)\left(\frac{M-1}{2}\right) + m\pi$$

where the $m\pi$ term accounts for possible sign changes in the amplitude of $X_w[k]$ as well as phase wrapping, so that $\measuredangle X_w[k]$ stays in the range $[-\pi, \pi]$.

From part (c) we know roughly where ω_0 should lie. Substituting $k = \hat{k}_0$ into the phase expression,

$$\begin{aligned} \measuredangle X_w[\hat{k}_0] &= \left(\omega_0 - \frac{2\pi \hat{k}_0}{N}\right)\left(\frac{M-1}{2}\right) + m\pi \\ &= (\omega_0 - \widehat{\omega}_0)\left(\frac{M-1}{2}\right) + m\pi \end{aligned}$$

The magnitude of the error $|\omega_0 - \widehat{\omega}_0|$ is bounded by π/N, so the first term lies within the range $[-\pi, \pi]$ even for the case $M = N$. In addition, $\widehat{\omega}_0$ lies within the main lobe of $X_w(e^{j\omega})$ bounded by $\omega_0 - \frac{2\pi}{M}$ and $\omega_0 + \frac{2\pi}{M}$, so the amplitude at $\omega = \widehat{\omega}_0$ is positive. We can therefore set $m = 0$ in the phase equation.

Solving the phase equation for ω_0 with $m = 0$,

$$\omega_0 = \widehat{\omega}_0 + \frac{2\measuredangle X_w[\hat{k}_0]}{M-1}$$

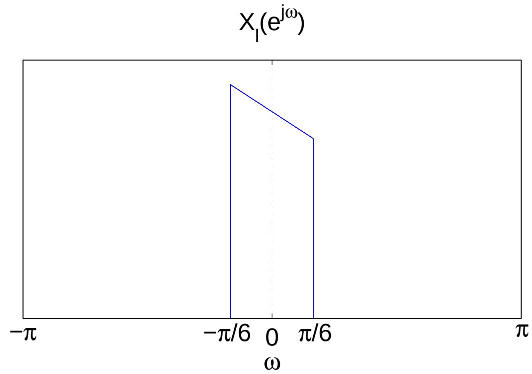
and $\Omega_0 = \frac{\omega_0}{T}$. We can obtain two estimates of Ω_0 for the two window choices $w_1[n]$ ($M = 32$) and $w_2[n]$ ($M = 8$), using the values of $\widehat{\omega}_0$ and $\hat{k}_0$ from part (c) in both cases, and check that they are consistent.

Problem 9.5 (OSB 10.44)

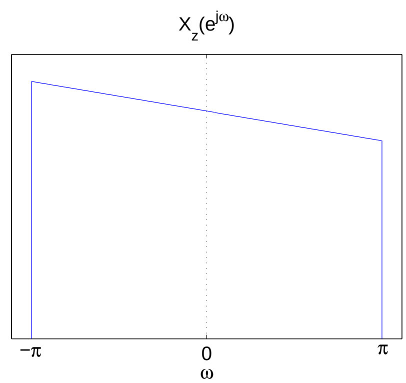
(a) After the lowpass filter, the highest frrequency in the signal is $\Delta\omega$. To avoid aliasing in the downsampler we must have:

$$\begin{aligned} M\Delta\omega &\leq \pi \\ M &\leq \frac{\pi}{\Delta\omega} \\ &\leq \frac{N}{2k_\Delta} \\ M_{\max} &= \frac{N}{2k_\Delta} \end{aligned}$$

(b) The Fourier transform of $x_l[n]$ looks like:



so $M = 6$ is the largest M we can use that avoids aliasing. The Fourier transform of $x_z[n]$ then looks like:



Taking the DFT of $x_z[n]$ gives us N samples of $X_z(e^{j\omega})$ spaced $\frac{2\pi}{N}$ apart in frequency. By comparing Figure P10.44-2 with the sketch of $X_z(e^{j\omega})$ above, we see that these samples are the desired samples of $X(e^{j\omega})$ spaced by $\frac{2\Delta\omega}{N}$ from $\omega_c - \Delta\omega$ to $\omega_c + \Delta\omega$.

Note that after downsampling the endpoints of the zoomed region overlap. Therefore, we cannot use the value $X_z\left[\frac{N}{2}\right]$ that corresponds to both $X_z(e^{j\pi})$ and $X_z(e^{-j\pi})$. However, we already have these frequency samples from the original N-point DFT $X_N[k]$, specifically from the values $X_N[k_c - k_\Delta]$ and $X_N[k_c + k_\Delta]$.

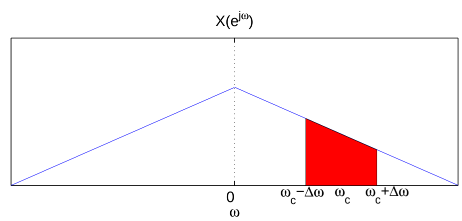
(c) The system $p[n]$ replicates $X_N[n]$ with a period of N to create $\widetilde{X}_N[n]$. The expander inserts $M - 1$ zeros in between each pair of samples in $\widetilde{X}_N[n]$. Thus, the samples $n = k_c - k_\Delta$ and $n = k_c + k_\Delta$ in $X_N[n]$ that border the zoom region map to $n = M(k_c - k_\Delta)$ and $n = M(k_c + k_\Delta)$. The system $h[n]$ interpolates between the nonzero points to fill in the "missing" samples. Since the Type I linear phase filter is of length 513, it adds a delay of $\frac{M}{2} = \frac{512}{2} = 256$ samples. The desired samples of $\widetilde{X}_{NM}[n]$ now lie in the region:

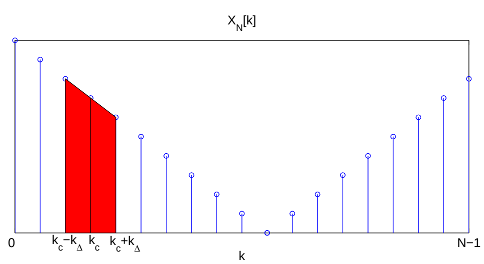
$$M(k_c - k_\Delta) + 256 \le n \le M(k_c + k_\Delta) + 256$$
$$k'_c - k'_\Delta \le n \le k'_c + k'_\Delta$$

where

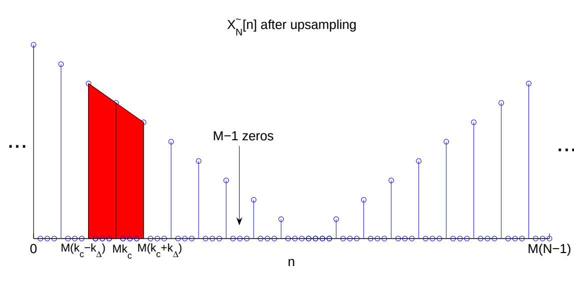
$$k'_c = Mk_c + 256$$
$$k'_\Delta = Mk_\Delta$$

(d) Sketches of $X(e^{j\omega})$ and $X_N[k]$ look like:

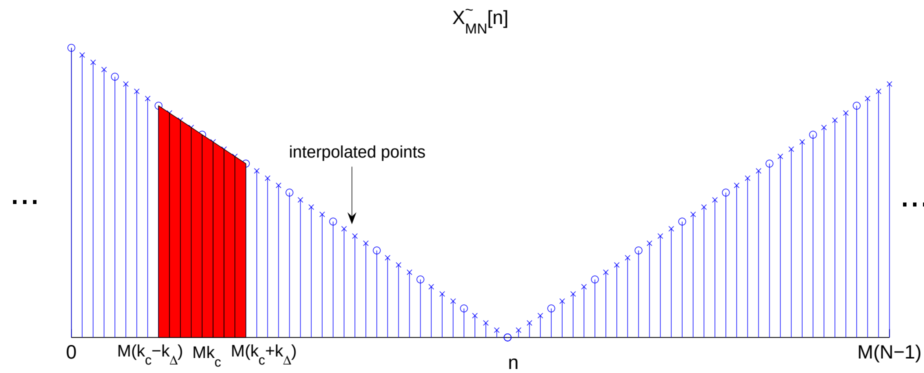




After periodically replicating and expanding by M we have:



Filtering by $h[n]$ interpolates between the non-zero samples. We assume that $h[n]$ is an ideal zero-phase filter in sketching $\widetilde{X}_{MN}[n]$. The interpolated points are marked with an $\times$.



Thus, we need to extract the points

$$M(k_c - k_\Delta) \leq n \leq M(k_c + k_\Delta)$$
$$k'_c - k'_\Delta \leq n \leq k'_c + k'_\Delta$$

where

$$k'_c = Mk_c$$
$$k'_\Delta = Mk_\Delta$$

Problem 9.6 (OSB 10.37)

(a) The autocorrelation estimate $\bar{\phi}[m]$ is the inverse Fourier transform of the averaged periodogram $\bar{I}(\omega)$:

$$\bar{I}(\omega) = \frac{1}{K}\sum_{r=0}^{K-1} I_r(\omega)$$

where $I_r(\omega)$ is the periodogram of the rth segment and K periodograms are averaged.

Substituting the above into the inverse Fourier transform and exchanging the order of averaging and integration,

$$\bar{\phi}[m] = \frac{1}{K}\sum_{r=0}^{K-1}\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi} I_r(\omega)e^{j\omega n}d\omega\right\}$$

We recognize the quantity in braces as the inverse Fourier transform of $I_r(\omega)$, which we denote as $i_r[m]$. $\bar{\phi}[m]$ is now expressed as the average of the sequences $i_r[m]$. Taking the expectation (a linear operation),

$$E\left\{\bar{\phi}[m]\right\} = \frac{1}{K}\sum_{r=0}^{K-1} E\left\{i_r[m]\right\}$$

We now need to find the expectations $E\{i_r[m]\}$.

From the definition of the periodogram:

$$I_r(\omega) = \frac{1}{LU}|X_r(e^{j\omega})|^2 = \frac{1}{LU}X_r(e^{j\omega})X_r(e^{j\omega})^*$$

For real $x_r[n]$, $X_r(e^{j\omega})^* = X_r(e^{-j\omega})$ corresponds to $x_r[-n]$, so $i_r[m]$ corresponds to $x_r[n]$ convolved with its time reversal, i.e. its aperiodic autocorrelation:

$$i_r[m] = \frac{1}{LU}\sum_{n=0}^{L-1} x_r[n]x_r[m+n]$$

Substituting the definition of $x_r[n]$ and taking the expectation,

$$\begin{aligned} E\{i_r[m]\} &= E\left\{\frac{1}{LU}\sum_{n=0}^{L-1} x[rR+n]w[n]x[rR+m+n]w[m+n]\right\}, \\ &= \frac{1}{LU}\sum_{n=0}^{L-1} w[n]w[m+n]E\{x[rR+n]x[m+rR+n]\}, \\ &= \frac{1}{LU}c_{ww}[m]\phi_{xx}[m] \end{aligned}$$

The expectation only acts upon the $x[\cdot]$ terms to produce $\phi_{xx}[m]$, and we identify the remaining sum as $c_{ww}[m]$, the autocorrelation of $w[n]$.

The expectation $E\{i_r[m]\}$ is now independent of r, so taking its average yields the same expression:

$$E\left\{\bar{\phi}[m]\right\} = \frac{1}{LU}c_{ww}[m]\phi_{xx}[m]$$

as desired.

(b) We follow a similar strategy as in part (a). First we express $\bar{\phi}_p[m]$ as an average:

$$\bar{\phi}_p[m] = \frac{1}{K}\sum_{r=0}^{K-1}\left\{\frac{1}{N}\sum_{k=0}^{N-1} I_r[k]e^{j(2\pi/N)km}\right\}, \quad m = 0, 1, \ldots, N-1$$

The quantity in braces is now the IDFT of $I_r[k]$, where $I_r[k]$ is the periodogram of the rth segment sampled at N evenly spaced frequencies. We denote this IDFT as $i_{rp}[m]$. As before,

$$E\left\{\bar{\phi}_p[m]\right\} = \frac{1}{K}\sum_{r=0}^{K-1} E\{i_{rp}[m]\}, \quad m = 0, 1, \ldots, N-1$$

From our knowledge of the DFT, the IDFT of samples of $I_r(\omega)$ is equal to $i_r[m]$ with possible time-aliasing:

$$i_{rp}[m] = \sum_{l=-\infty}^{\infty} i_r[m-lN], \quad m = 0, 1, \ldots, N-1$$

Taking the expectation and substituting for $E\{i_r[m]\}$ from part (a):

$$E\{i_{rp}[m]\} = \frac{1}{LU}\sum_{l=-\infty}^{\infty} c_{ww}[m-lN]\phi_{xx}[m-lN], \quad m = 0, 1, \ldots, N-1$$

which is again independent of r. Therefore

$$E\left\{\bar{\phi}_p[m]\right\} = \frac{1}{LU}\sum_{l=-\infty}^{\infty} c_{ww}[m-lN]\phi_{xx}[m-lN], \quad m = 0, \ldots, N-1$$

(c) $E\left\{\bar{\phi}_p[m]\right\} = E\left\{\bar{\phi}[m]\right\}$, $m = 0, 1, \ldots, L-1$ when the periodic replication of $i_r[m]$ to produce $i_{rp}[m]$ is non-overlapping. We know that $i_r[m]$ is the autocorrelation of $x_r[n]$, a finite segment of length L, so $i_r[m]$ has length $2L-1$ and extends from $m = -L+1$ to $m = L-1$. If the period of replication N is at least $2L-1$, then copies of $i_r[m]$ will not overlap in $i_{rp}[m]$ and we will achieve the desired equality. Therefore $N \geq 2L-1$.

Problem 9.7

(a) **Answer:** 1D, 2A, 3B, 4C.

We first consider the main lobe width of the sinusoidal component. The width of the main lobe is inversely proportional to the length of the window applied, i.e. the number of actual data points (before zero-padding) included in each periodogram. Figure B clearly has the widest main lobe, so it corresponds to the 32-point DFTs in method 3. This fixes 3B.

The remaining estimates A, C, and D can be distinguished by their variances. C has the least variance so it was produced with the most averaging. This fixes 4C.

A and D can be recognized as depicting samples of the same DTFT, with the sampling in A being more dense because of zero-padding.

(b) **Answer:** 1C, 2B, 3A, 4D.

From the shapes and smoothness (variances) of the estimates, we identify B and C as estimates based on periodograms, and A and D as estimates based on all-pole models. (Implicit in this statement is the knowledge that the all-pole models have orders of at most 16. An estimate obtained from a much higher-order all-pole model could look more like a periodogram estimate.)

Between A and D, D is produced by a higher-order model because it has more relative maxima and minima. Note also that the two sinusoidal components can be distinguished using the 16th-order model, while the 8th-order model struggles to show a second sinusoidal component.

Between B and C, B has lower variance and is produced by Welch's method which involves averaging.