Massachusetts Institute of Technology
Department of Electrical Engineering and Computer Science

6.341: Discrete-Time Signal Processing

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Problem Set 7 Solutions

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Problem 7.1

(a) The normal (Yule-Walker) equations are:

$$\phi_s[i] = \sum_{k=1}^{2} a_k \phi_s[i-k], \quad i = 1, 2,$$

or in matrix form:

$$\begin{bmatrix} \phi_s[0] & \phi_s[1] \\ \phi_s[1] & \phi_s[0] \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} \phi_s[1] \\ \phi_s[2] \end{bmatrix}.$$

(b) Let $s_1[n] = (\frac{1}{3})^n u[n]$ and $s_2[n] = (-\frac{1}{2})^n u[n]$. We calculate the following auto- and cross-correlations for $m > 0$,

$$\phi_{s_1}[m] = \sum_{n=-\infty}^{\infty} s_1[n+m]s_1[n] = \frac{9}{8}\left(\frac{1}{3}\right)^m$$

$$\phi_{s_2}[m] = \sum_{n=-\infty}^{\infty} s_2[n+m]s_2[n] = \frac{4}{3}\left(-\frac{1}{2}\right)^m$$

$$\phi_{s_1 s_2}[m] = \sum_{n=-\infty}^{\infty} s_1[n+m]s_2[n] = \frac{6}{7}\left(\frac{1}{3}\right)^m$$

$$\phi_{s_2 s_1}[m] = \sum_{n=-\infty}^{\infty} s_2[n+m]s_1[n] = \frac{6}{7}\left(-\frac{1}{2}\right)^m.$$

Since

$$\phi_s[m] = \phi_{s_1}[m] + \phi_{s_2}[m] + \phi_{s_1 s_2}[m] + \phi_{s_2 s_1}[m]$$

and $\phi_s[m]$ is an even function of m, we sum the four correlations and replace m by $|m|$:

$$\phi_s[m] = \frac{111}{56}\left(\frac{1}{3}\right)^{|m|} + \frac{46}{21}\left(-\frac{1}{2}\right)^{|m|}.$$

Note that the cross-correlations $\phi_{s_1 s_2}[m]$ and $\phi_{s_2 s_1}[m]$ by themselves are not even.

So $\phi_s[0] = 4.17$, $\phi_s[1] = -.4345$ and $\phi_s[2] = .7678$.

(c) Substituting the values of $\phi_s[i]$ into the normal equations and solving for the a_i's results in $a_1 = -0.0859, a_2 = .1751$.

(d) The normal (Yule-Walker) equations are:

$$\phi_s[i] = \sum_{k=1}^{3} a_k \phi_s[i-k], \quad i = 1, 2, 3,$$

or in matrix form:

$$\begin{bmatrix} \phi_s[0] & \phi_s[1] & \phi_s[2] \\ \phi_s[1] & \phi_s[0] & \phi_s[1] \\ \phi_s[2] & \phi_s[1] & \phi_s[0] \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} \phi_s[1] \\ \phi_s[2] \\ \phi_s[3] \end{bmatrix}.$$

(e) $\phi_s[3] = -.2004$.

(f) Substituting the values of $\phi_s[i]$ into the normal equations and solving for the a_i's results in $a_1 = -0.0833, a_2 = 0.1738, a_3 = -0.0146$.

(g) Yes. The signal $s[n]$ is NOT the impulse response of an all-pole filter. Increasing the order will in general update all previous coefficients in an attempt to model $s[n]$ more accurately.

(h) In problem 6.7 $s[n]$ was the impulse response of a two-pole system, which we could model perfectly using a two-pole model. Increasing the order beyond $p = 2$ achieves nothing. In this problem $s[n]$ does not arise from an all-pole system, so it is not generally possible to perfectly model $s[n]$ using only poles. Nevertheless, increasing the order of the all-pole model will yield a closer and closer approximation.

(i) The difference equation for which the impulse response is $s[n]$ is:

$$s[n] = -\frac{1}{6}s[n-1] + \frac{1}{6}s[n-2] + 2\delta[n] + \frac{1}{6}\delta[n-1].$$

For $n \geq 2$ the impulses are zero:

$$s[n] = -\frac{1}{6}s[n-1] + \frac{1}{6}s[n-2].$$

Thus the linear prediction coefficients are $a_1 = -1/6, a_2 = 1/6$.

Problem 7.2 (OSB 8.31)

We re-write the desired samples of $X(z)$ in terms of the DFT of a second sequence $x_1[n]$. $x[n]$ is only non-zero for $0 \leq n \leq 9$:

$$\begin{aligned}
X(z) &= \sum_{n=0}^{9} x[n] z^{-n} \\
X(z)\mid_{z=0.5e^{j[(2\pi k/10)+(\pi/10)]}} &= \sum_{n=0}^{9} x[n] \left(0.5e^{j[(2\pi k/10)+(\pi/10)]}\right)^{-n} \\
&= \sum_{n=0}^{9} x[n] \left(0.5e^{j\pi/10}\right)^{-n} e^{-j(2\pi/10)kn} \\
&= \sum_{n=0}^{9} x_1[n] e^{-j(2\pi/10)kn} \\
&= X_1[k],\ k = 0, 1, \ldots, 9
\end{aligned}$$

where we have defined $x_1[n] = \left(2e^{-j\pi/10}\right)^n x[n]$ and we recognize the second last line as the 10-point DFT of $x_1[n]$.

Thus $x_1[n] = \left(2e^{-j\pi/10}\right)^n x[n]$.

Problem 7.3 (OSB 8.32)

Answer: (c)

Since $y[n]$ is $x[n]$ expanded by 2, the DTFT $Y(e^{j\omega})$ is equal to $X(e^{2j\omega})$, i.e. $X(e^{j\omega})$ with the frequency axis compressed by a factor of 2. The 16-point DFT $Y[k]$ samples $Y(e^{j\omega})$ at frequencies $\omega = \frac{2\pi k}{16},\ k = 0, 1, \ldots, 15$, which is equivalent to sampling $X(e^{j\omega})$ at frequencies $\omega = \frac{2\pi k}{8},\ k = 0, 1, \ldots, 15$. But since $X(e^{j\omega})$ is periodic with period 2π, the last eight samples are the same as the first eight, which in turn are equal to the 8-point DFT $X[k]$. In other words, $Y[k]$ samples $X(e^{j\omega})$ from 0 to 4π instead of from 0 to 2π. Therefore $Y[k]$ is equal to $X[k]$ repeated back-to-back.

Problem 7.4 (OSB 8.37)

- For $g_1[n]$, choose $H_7[k]$.

  We can think of this as a time reversal followed by a shift by $-N+1$.

$$\begin{aligned}
G_1[k] &= \sum_{i=0}^{N-1} g_1[i] W_N^{ik} \qquad\qquad k = 0, \cdots, N-1 \\
&= \sum_{i=0}^{N-1} x[N-1-i] W_N^{ik} \\
&= \sum_{j=0}^{N-1} x[j] W_N^{k(N-1-j)} \\
&= W_N^{k(N-1)} \sum_{j=0}^{N-1} x[j] W_N^{(-k)j} \\
&= W_N^{-k} X[((-k))_N] \\
&= e^{j2\pi k/N} X(e^{-j2\pi k/N})
\end{aligned}$$

- For $g_2[n]$, choose $H_8[k]$.

  This is modulation in time by $(-1)^n = e^{j\pi n}$, or a shift in the frequency domain by π.

$$\begin{aligned}
G_2[k] &= \sum_{i=0}^{N-1} g_2[i] W_N^{ik} \qquad\qquad k = 0, \cdots, N-1 \\
&= \sum_{i=0}^{N-1} (-1)^i x[i] W_N^{ik} \\
&= \sum_{i=0}^{N-1} x[i] W_N^{i(k+N/2)} \\
&= X[((k+N/2))_N] \\
&= X(e^{j(2\pi/N)(k+N/2)})
\end{aligned}$$

- For $g_3[n]$, choose $H_3[k]$.

  We can interpret the DFT $X[k]$ as the Fourier series coefficients of $\tilde{x}[n]$, the periodic replication of $x[n]$ with period N. Given this interpretation, the DFT $G_3[k]$ is also equal to the Fourier series of $\tilde{x}[n]$, but considered as having a period of $2N$. However, since $\tilde{x}[n]$ has a fundamental period of N, the even-indexed coefficients of the length $2N$ Fourier series will correspond to the length N Fourier series coefficients (i.e. $X[k]$), while the odd-indexed coefficients will be zero because they are not necessary.

$$
\begin{aligned}
G_3[k] &= \sum_{i=0}^{2N-1} g_3[i]W_{2N}^{ik} \qquad\qquad k = 0, \cdots, 2N-1 \\
&= \sum_{i=0}^{N-1} x[i]W_{2N}^{ik} + \sum_{i=N}^{2N-1} x[i-N]W_{2N}^{ik} \\
&= \sum_{i=0}^{N-1} x[i](W_{2N}^{ik} + W_{2N}^{(i+N)k}) \\
&= \sum_{i=0}^{N-1} x[i]W_{2N}^{ik}(1 + W_{2N}^{Nk}) \\
&= \sum_{i=0}^{N-1} x[i]W_{2N}^{ik}(1 + (-1)^k) \\
&= X(e^{j2\pi k/2N})(1 + (-1)^k) \\
&= \begin{cases} 2X(e^{j2\pi k/2N}), & k \text{ even} \\ 0. & k \text{ odd} \end{cases}
\end{aligned}
$$

- For $g_4[n]$, choose $H_6[k]$.

  The DFT of $g_4[n]$ is equal to the DFS of $\tilde{x}[n]$, the periodic replication of $x[n]$ with a period of $N/2$. In other words, $g_4[n]$ is $x[n]$ aliased in time. The DFS of $\tilde{x}[n]$ is in turn equal to samples of $X(e^{j\omega})$ spaced by $\frac{2\pi}{N/2} = \frac{4\pi}{N}$.

$$
\begin{aligned}
G_4[k] &= \sum_{i=0}^{N/2-1} g_4[i] W_{N/2}^{ik} \qquad\qquad k = 0, \cdots, N/2-1 \\
&= \sum_{i=0}^{N/2-1} (x[i] + x[i+N/2]) W_{N/2}^{ik} \\
&= \sum_{i=0}^{N/2-1} x[i] W_{N/2}^{ik} + \sum_{i=0}^{N/2-1} x[i+N/2] W_{N/2}^{ik} \\
&= \sum_{i=0}^{N/2-1} x[i] W_{N/2}^{ik} + \sum_{i=0}^{N/2-1} x[i+N/2] W_{N/2}^{k(i+N/2)} \\
&= \sum_{i=0}^{N/2-1} x[i] W_{N/2}^{ik} + \sum_{j=N/2}^{N-1} x[j]) W_{N/2}^{jk} \\
&= \sum_{i=0}^{N-1} x[i] W_{N/2}^{ik} \\
&= \sum_{i=0}^{N-1} x[i] (e^{-j(4\pi/N)ik}) \\
&= X(e^{j4\pi k/N})
\end{aligned}
$$

- For $g_5[n]$, choose $H_2[k]$.

  We are increasing the length of the signal by zero padding. Thus, we are taking more closely spaced samples of $X(e^{j\omega})$.

$$
\begin{aligned}
G_5[k] &= \sum_{i=0}^{2N-1} g_5[i] W_{2N}^{ik} \qquad\qquad k = 0, \cdots, 2N-1 \\
&= \sum_{i=0}^{N-1} x[i] W_{2N}^{ik} \\
&= \sum_{i=0}^{N-1} x[i] W_{N}^{i(k/2)} \\
&= X(e^{j2\pi(k/2)/N}) \\
&= X(e^{j2\pi k/(2N)})
\end{aligned}
$$

- For $g_6[n]$, choose $H_1[k]$.

  We are expanding $x[n]$ by 2 to form $g_6[n]$. The DTFT of $g_6[n]$ is equal to $X(e^{2j\omega})$, i.e. $X(e^{j\omega})$ with the frequency axis compressed by 2. The $2N$ values of $G_6[k]$ sample two periods of $X(e^{j\omega})$, so the last N samples are equal to the first N. Moreover, the first N samples are the same as those in $X[k]$. Thus $G_6[k]$ contains the same frequency samples at $\omega = \frac{2\pi k}{N}$, but now k ranges from 0 to $2N-1$.

$$\begin{aligned}
G_6[k] &= \sum_{i=0}^{2N-1} g_6[i] W_{2N}^{ik} \qquad\qquad k = 0, \cdots, 2N-1 \\
&= \sum_{i=0}^{N-1} g[2i] W_{2N}^{2ik} + \sum_{i=0}^{N-1} g[2i+1] W_{2N}^{(2i+1)k} \\
&= \sum_{i=0}^{N-1} x[i] W_N^{ik} + 0 \\
&= X(e^{j2\pi k/N})
\end{aligned}$$

- For $g_7[n]$, choose $H_5[k]$.

  We are decimating $x[n]$ by 2, so $X(e^{j\omega})$ is vertically scaled by $\frac{1}{2}$, horizontally stretched by 2, and replicated once. We then obtain samples of the resulting DTFT at frequencies $\omega = \frac{2\pi}{N/2}$.

$$\begin{aligned}
G_7[k] &= \sum_{i=0}^{N/2-1} g_7[i] W_{N/2}^{ik} \qquad\qquad k = 0, \ldots, N/2-1 \\
&= \sum_{i=0}^{N/2-1} x[2i] W_{N/2}^{ik} \\
&= \sum_{i=0}^{N/2-1} x[2i] W_N^{(2i)k} \\
&= \sum_{i=0,\, i \text{ even}}^{N-1} x[i] W_N^{ik} \\
&= \sum_{i=0}^{N-1} \frac{1}{2} \left(x[i] + (-1)^i x[i] \right) W_N^{ik} \\
&= \frac{1}{2} \left\{ X[k] + X[((k+N/2))_N] \right\} \\
&= 0.5 \left\{ X(e^{j2\pi k/N}) + X(e^{j2\pi(k+N/2)/N}) \right\}
\end{aligned}$$

Problem 7.5 (OSB 8.46)

In general, (i) holds if the periodic replication of $x_i[n]$ is even symmetric about $n = 0$; (ii) holds if $x_i[n]$ has some point of symmetry; (iii) holds if the periodic replication of $x_i[n]$ has some point of symmetry. Note the subtle difference between (ii) and (iii).

- For $x_1[n]$:

$$\begin{aligned} X_1[k] &= 3(1 + W_5^{4k}) + 1(W_5^k + W_5^{3k}) + 2(W_5^{2k}) \\ &= 2W_5^{2k}\{3\cos(2k(2\pi/5)) + 1\cos(k(2\pi/5)) + 1\} \\ X_1(e^{j\omega}) &= 2e^{-j2\omega}\{3\cos(2\omega) + \cos\omega + 1\} \end{aligned}$$

  (i) No, $X_1[k]$ is not real for all k.
  (ii) Yes, $X_1(e^{j\omega})$ has generalized linear phase.
  (iii) Yes.

- For $x_2[n]$:

$$\begin{aligned} X_2(e^{j\omega}) &= 3 + 2e^{-j2.5\omega}\{1\cos(1.5\omega) + 2\cos(0.5\omega)\} \\ X_2[k] &= 3 + 2W_5^{2.5k}\{\cos(1.5k(2\pi/5)) + 2\cos(0.5k(2\pi/5))\} \\ &= 3 + 2(-1)^k\{1\cos(1.5k(2\pi/5)) + 2\cos(0.5k(2\pi/5))\} \end{aligned}$$

  (i) Yes.
  (ii) No.
  (iii) Yes.

- For $x_3[n]$:

$$\begin{aligned} X_3(e^{j\omega}) &= 1 + 2e^{-j2\omega}\{2\cos(2\omega) + 1\cos(1\omega) + 1\} \\ X_3[k] &= 1 + 2W_5^{2k}\{2\cos(2k(2\pi/5)) + 1\cos(k(2\pi/5)) + 1\} \end{aligned}$$

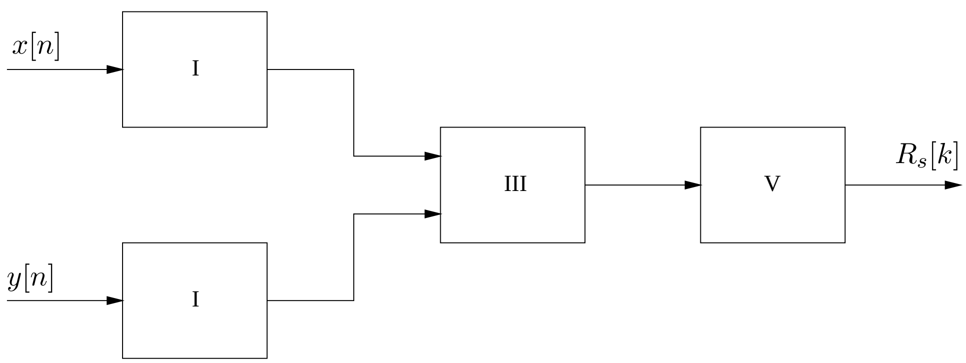
  (i) No.
  (ii) No.
  (iii) No.

Problem 7.6 (OSB 8.59)

We want to compute $R_s[k] = R(e^{j2\pi k/128})$, the DTFT of $r[n]$ sampled at 128 equally spaced frequencies.

Both $x[n]$ and $y[n]$ are signals of length 256, so their linear convolution $r[n]$ has length 511. If we had $r[n]$, we could calculate $R_s[k]$ by time-aliasing $r[n]$ to 128 samples (periodically replicating $r[n]$ with a period of 128 and extracting one period) and taking the 128-point DFT (module V). However, a linear convolution module is not available, so an alternative way of time-aliasing $r[n]$ is through circular convolution of $x[n]$ and $y[n]$. $x[n]$ and $y[n]$ can be circularly convolved by periodically replicating both signals with a period of 128 using module I, performing periodic convolution using module III, and extracting one period of the periodic convolution. The result of this circular convolution is equal to $r[n]$ time-aliased to 128 samples. However, since the 128-point DFT module (module V) only considers its input between $n = 0$ and $n = 127$, the explicit extraction of one period is not necessary.

The implementation just described is pictured below. The total cost is 110 units.



Problem 7.7

(a) Assuming that the overlap-save method is correctly implemented, the output $y[n]$ of S can be represented as the *linear* convolution $y[n] = x[n] * h[n]$. The impulse response $h[n]$ corresponding to $H[k]$ is a finite sequence of length 256. However, an ideal frequency-selective filter has an infinite impulse response. Therefore, S cannot be an ideal frequency-selective filter.

(b) The impulse response $h[n]$ of S is the IDFT of $H[k]$. Since $H[k]$ is real and even in the circular sense ($H[k] = H[((-k))_{256}]$), $h[n]$ is real.

(c)

$$
\begin{aligned}
h[n] &= \frac{1}{256}\sum_{k=0}^{255} H[k]W_{256}^{-kn} \qquad\qquad 0 \le n \le 255 \\
&= \frac{1}{256}\sum_{k=0}^{31} W_{256}^{-kn} + \frac{1}{256}\sum_{k=225}^{255} W_{256}^{-n(k-256)} \\
&= \frac{1}{256}\sum_{k=0}^{31} W_{256}^{-kn} + \frac{1}{256}\sum_{k=-31}^{-1} W_{256}^{-kn} \\
&= \frac{1}{256}\sum_{k=-31}^{31} W_{256}^{-kn} \\
&= \frac{1}{256}\frac{W_{256}^{31n} - W_{256}^{-32n}}{1 - W_{256}^{-n}} \\
&= \frac{1}{256}\frac{W_{256}^{-0.5n}\left(W_{256}^{31.5n} - W_{256}^{-31.5n}\right)}{W_{256}^{-0.5n}\left(W_{256}^{0.5n} - W_{256}^{-0.5n}\right)} \\
&= \frac{\sin\frac{63\pi n}{256}}{256\sin\frac{\pi n}{256}}
\end{aligned}
$$

In sum,

$$
h[n] = \begin{cases} \dfrac{\sin\dfrac{63\pi n}{256}}{256\sin\dfrac{\pi n}{256}} & 0 \le n \le 255 \\ 0 & \text{otherwise} \end{cases}
$$