Massachusetts Institute of Technology
Department of Electrical Engineering and Computer Science

6.341: Discrete-Time Signal Processing

Fall 2005

Solutions for Problem Set 3

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Problem 3.1

No, it is not necessarily causal. As a counterexample consider the case of a non-integer delay such as $H(e^{j\omega}) = e^{-j\frac{\omega}{2}}$, where $\tau(\omega) = \frac{1}{2}$ for all ω but the corresponding impulse response $h[n]$ is certainly noncausal.

Problem 3.2

(a) For odd n, $x[n] = 0$ and $R_{xx}[n,m] = 0$. For even n and odd m, $x[n+m] = 0$ and again $R_{xx}[n,m] = 0$. When both n and m are even: $\mathcal{E}[x[n]x[n+m]] = \mathcal{E}[w[\frac{n}{2}]w[\frac{n+m}{2}]] = \sigma_w^2\delta[m]$. $x[n]$ is not WSS.

(b)

$$
\begin{aligned}
R_{yy}[n,m] &= \mathcal{E}[(\sum_{k_1'=-\infty}^{\infty} x[k_1']h[n-k_1'])(\sum_{k_2'=-\infty}^{\infty} x[k_2']h[n+m-k_2'])] \\
&= \mathcal{E}[(\sum_{k_1=-\infty}^{\infty} x[2k_1]h[n-2k_1])(\sum_{k_2=-\infty}^{\infty} x[2k_2]h[n+m-2k_2])] \\
&= \sum_{k_1=-\infty}^{\infty}\sum_{k_2=-\infty}^{\infty} h[n-2k_1]h[n+m-2k_2]\mathcal{E}[x[2k_2]x[2k_1]] \\
&= \sum_{k_1=-\infty}^{\infty}\sum_{k_2=-\infty}^{\infty} h[n-2k_1]h[n+m-2k_2]\sigma_w^2\delta[k_2-k_1] \\
&= \sigma_w^2 \sum_{k_1=-\infty}^{\infty} h[n-2k_1]h[n+m-2k_1] \\
&= \sigma_w^2 \sum_{k=-\infty}^{\infty} h[-2k]h[m-2k]
\end{aligned}
$$

Where we made the substitution $k = k_1 - \frac{n}{2}$ for the last step.

(c) $\mathcal{E}[d[n]] = \mathcal{E}[y[2n]] = \mathcal{E}[\sum_{k=-\infty}^{\infty} x[k]h[2n-k]] = \sum_{k=-\infty}^{\infty}\mathcal{E}[x[k]]h[2n-k] = 0$, so the mean value of $d[n]$ is independent of the time index n.

Since $R_{dd}[n,m] = R_{yy}[2n,2m]$ we need to examine $R_{yy}[2n,2m]$ for the case where both n and m are even. Our answer in (b) does not depend on n, so $d[n]$ will be wide sense stationary as long as the sum over k converges. It is enough to require that: $\sum_{k=-\infty}^{\infty} |h[2k]|^2 < \infty$

Problem 3.3

OSB Problem 4.47

(a)

$$\begin{aligned}\phi_{xx}[m] &= E(x[n]x^*[n+m]) \\ &= E[x_c(nT)x_c^*(nT+mT)] \\ &= \phi_{x_cx_c}(mT)\end{aligned}$$

(b) Since $\phi_{xx}[m]$ is samples of $\phi_{x_cx_c}(\tau)$;

$$P_{xx}(e^{j\omega}) = \frac{1}{T}\sum_{k=-\infty}^{+\infty} P_{x_cx_c}\left(\frac{\omega}{T} + \frac{2\pi k}{T}\right)$$

(c) If $P_{x_cx_c}(\Omega) = 0$, for $|\Omega| \geq \pi/T$.

Problem 3.4

(a) For uniform distribution:

$$\begin{aligned}\mathcal{E}(e[n]) &= \textstyle\int_{-\Delta/2}^{\Delta/2} e\frac{1}{\Delta}de = 0 \\ \sigma_e^2 &= \mathcal{E}(e^2[n]) = \textstyle\int_{-\Delta/2}^{\Delta/2} e^2\frac{1}{\Delta}de = \frac{\Delta^2}{12} \\ \mathcal{E}(e[n]e[n+m]) &= \mathcal{E}(e^2[n])\delta[m] = \frac{\Delta^2}{12}\delta[m]\end{aligned}$$

(b)

$$\sigma_x^2/\sigma_e^2 = 12\sigma_x^2/\Delta^2$$

(c) For given filter, we have

$$h[n] = \sum_{k=0}^{\infty} a^{2k}\delta[n-2k].$$

Thus,

$$
\begin{aligned}
H(e^{j\omega}) &= 1/(1-a^2e^{-2j\omega}) \\
\Phi_e(e^{j\omega}) &= \frac{\Delta^2}{12} \\
\Phi_{y_ey_e}(e^{j\omega}) &= \Phi_e(e^{j\omega})|H(e^{j\omega})|^2 \\
&= \frac{\Delta^2}{12|1-a^2e^{-2j\omega}|^2} \\
&= \frac{\Delta^2}{12(1+a^4-2a^2\cos 2\omega)} \\
&= \frac{\Delta^2}{12(1-a^2e^{-2j\omega})(1-a^2e^{2j\omega})} \\
\phi_{y_ey_e}[n] &= h[n]*h[-n] \\
&= \frac{\Delta^2}{12(1-a^4)}a^{2|n|} \text{ for even } n, \text{ 0 for odd } n \\
\sigma^2_{y_e} &= \phi y_e y_e[0] = \sigma_e^2 \sum_{k=-\infty}^{\infty} h^2[k] = \frac{\Delta^2}{12(1-a^4)}
\end{aligned}
$$

The variance of $x[n]$ is also scaled by the the same factor, so the SNR at the output is still $12\sigma_x^2/\Delta^2$.

Problem 3.5

(a) Since this is an LTI model, we can suppress e to find the contribution that x makes on y. Writing node equations:

$$
\begin{aligned}
D_1(z) &= X(z) - Y(z) \\
D_2(z) &= H_1(z)D_1(z) - Y(z) \\
Y(z) &= H_2(z)D_2(z)
\end{aligned}
$$

These can be solved by substitution to yield:

$$\frac{Y(z)}{X(z)} = z^{-1}$$

Similarly, we can suppress x to find the contribution that e makes on y. Writing node equations:

$$
\begin{aligned}
D_2(z) &= -H_1(z)Y(z) - Y(z) \\
Y(z) &= H_2(z)D_2(z) + E(z)
\end{aligned}
$$

These can be solved to yield:

$$\frac{Y(z)}{E(z)} = \left(1 - z^{-1}\right)^2$$

Therefore,

$$Y(z) = z^{-1}X(z) + \left(1 - z^{-1}\right)^2 E(z)$$

The inverse transform is: $y[n] = x[n-1] + f[n]$, where $f[n] = e[n] - 2e[n-1] + e[n-2]$.

(b)

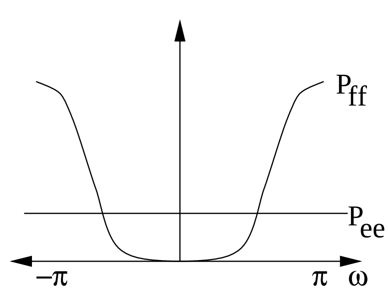
$$\begin{aligned} P_{ff}(e^{j\omega}) &= |H_{ey}(e^{j\omega})|^2 P_{ee}(e^{j\omega}) \\ &= \sigma_e^2 |(1 - e^{-j\omega})^2|^2 \\ &= \sigma_e^2 (1 - e^{-j\omega})^2 (1 - e^{j\omega})^2 \\ &= \sigma_e^2 (2 - 2\cos(\omega))^2 \\ &= \sigma_e^2 \left(4 \sin^2\left(\frac{\omega}{2}\right)\right)^2 \\ &= 16\sigma_e^2 \sin^4\left(\frac{\omega}{2}\right) \end{aligned}$$

$$\begin{aligned} \sigma_f^2 = \phi_{ff}[0] &= \frac{1}{2\pi}\int_{\omega=-\pi}^{\pi} P_{ff}(e^{j\omega}) d\omega \\ &= \frac{1}{2\pi}\int_{\omega=-\pi}^{\pi} 16\sigma_e^2 \sin^4\left(\frac{\omega}{2}\right) d\omega \\ &= 6\sigma_e^2 \end{aligned}$$

Alternatively, we could have found the total noise power as follows:
The total noise power σ_f^2 is the autocorrelation of $f[n]$ evaluated at 0:

$$\begin{aligned} \sigma_f^2 &= E[(e[n] - 2e[n-1] + e[n-2])^2] \\ &= E[e^2[n]] + E[(-2)^2 e^2[n-1]] + E[e^2[n-2]] \\ &= 6\sigma_e^2, \end{aligned}$$

where we have used linearity of expectations, and the fact that since $e[n]$ is white, $E[e[n]e[n-k]] = 0$ for $k \neq 0$.



(c) Since $X(e^{j\omega})$ is bandlimited between $\pm\pi/M$, when $y[n]$ is passed through $H_3(e^{j\omega})$ all of the information in $X(e^{j\omega})$ is preserved, so $w[n] = x[n-1] + g[n]$, where $g[n]$ is the contribution from the noise. Since the noise power spectral density was shaped to be very low for low frequencies, and high frequencies are cut off by $H_3(e^{j\omega})$, the variance of $g[n]$ is small, as explored below.

(d)

$$\begin{aligned}
\sigma_g^2 = \phi_{gg}[0] &= \frac{1}{2\pi}\int_{\omega=-\pi/M}^{\pi/M} P_{gg}(e^{j\omega})d\omega \\
&= \frac{1}{2\pi}\int_{\omega=-\pi/M}^{\pi/M} 16\sigma_e^2 \sin^4\left(\frac{\omega}{2}\right) d\omega \\
&\approx \frac{1}{2\pi}\int_{\omega=-\pi/M}^{\pi/M} \sigma_e^2\omega^4 d\omega \\
&= \frac{\sigma_e^2\pi^4}{5M^5}
\end{aligned}$$

(e) $X(j\Omega)$ must be bandlimited between $\pm\pi/(MT)$. $v[n] = x_c((Mn-1)T) + q[n]$. $\sigma_q^2 = \sigma_g^2$. The power spectrum of the noise at the output is $P_{qq}(e^{j\omega}) = \frac{1}{M}P_{gg}(e^{j\frac{\omega}{M}}) = (16/M)\sigma_e^2 \sin^4\left(\frac{\omega}{2M}\right)$.

The plot below uses an example of $M = 4$: