Massachusetts Institute of Technology
Department of Electrical Engineering and Computer Science

6.341: Discrete-Time Signal Processing

Fall 2005

Problem Set 1 Solutions

Issued: Thursday, September 8, 2005

Problem 1.1 (OSB 2.1)

Answers are in the back of the book but have a few typos:

(d) The system is causal when $n_0 \geq 0$, not when $n_0 \leq 0$.

(h) (not assigned but for your benefit) The system is also causal.

Problem 1.2 (OSB 2.6)

(a)

$$y[n] - \frac{1}{2}y[n-1] = x[n] + 2x[n-1] + x[n-2]$$

$$Y(e^{j\omega})\left[1 - \frac{1}{2}e^{-j\omega}\right] = X(e^{j\omega})\left[1 + 2e^{-j\omega} + e^{-j2\omega}\right]$$

$$H\left(e^{j\omega}\right) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{1 + 2e^{-j\omega} + e^{-j2\omega}}{1 - \frac{1}{2}e^{-j\omega}}$$

(b)

$$H(e^{j\omega}) = \frac{1 - \frac{1}{2}e^{-j\omega} + e^{-j3\omega}}{1 + \frac{1}{2}e^{-j\omega} + \frac{3}{4}e^{-j2\omega}} = \frac{Y(e^{j\omega})}{X(e^{j\omega})}$$

$$Y(e^{j\omega})\left[1 + \frac{1}{2}e^{-j\omega} + \frac{3}{4}e^{-j2\omega}\right] = X(e^{j\omega})\left[1 - \frac{1}{2}e^{-j\omega} + e^{-j3\omega}\right]$$

$$y[n] + \frac{1}{2}y[n-1] + \frac{3}{4}y[n-2] = x[n] - \frac{1}{2}x[n-1] + x[n-3]$$

Problem 1.3 (OSB 2.11)

We can write $x[n]$ as a sum of exponentials and compute the response of the system to each exponential:

For $x[n] = \sin\left(\frac{\pi n}{4}\right)$,

$$x[n] = \sin\left(\frac{\pi n}{4}\right) = \frac{1}{2j}\left[e^{j\frac{\pi}{4}n} - e^{-j\frac{\pi}{4}n}\right]$$

Response to $e^{j\frac{\pi}{4}n}$:

$$H(e^{j\frac{\pi}{4}})e^{j\frac{\pi}{4}n} = \left[\frac{1 - e^{-j2\frac{\pi}{4}}}{1 + \frac{1}{2}e^{-j4\frac{\pi}{4}}}\right] e^{j\frac{\pi}{4}n} = 2(1+j)e^{j\frac{\pi}{4}n} = 2\sqrt{2}e^{j\frac{\pi}{4}}e^{j\frac{\pi}{4}n}$$

Response to $e^{-j\frac{\pi}{4}n}$:

$$H(e^{-j\frac{\pi}{4}})e^{-j\frac{\pi}{4}n} = \left[\frac{1 - e^{j2\frac{\pi}{4}}}{1 + \frac{1}{2}e^{j4\frac{\pi}{4}}}\right] e^{-j\frac{\pi}{4}n} = 2(1-j)e^{-j\frac{\pi}{4}n} = 2\sqrt{2}e^{-j\frac{\pi}{4}}e^{-j\frac{\pi}{4}n}$$

$$y[n] = \frac{1}{2j}\left[H(e^{j\frac{\pi}{4}})e^{j\frac{\pi}{4}n} - H(e^{-j\frac{\pi}{4}})e^{-j\frac{\pi}{4}n}\right] = \frac{1}{2j}\left[2\sqrt{2}\left[e^{j\left(\frac{\pi}{4}n+\frac{\pi}{4}\right)} - e^{-j\left(\frac{\pi}{4}n+\frac{\pi}{4}\right)}\right]\right]$$

$$y[n] = 2\sqrt{2}\sin\left(\frac{\pi(n+1)}{4}\right)$$

Note: Answer in the back of the book has a typo.

For $x[n] = \sin\left(\frac{7\pi n}{4}\right)$, we have to map the frequency $\frac{7\pi}{4}$ into the $-\pi$ to π range where $H(e^{j\omega})$ is defined.

$$x[n] = \sin\left(\frac{7\pi n}{4}\right) = \sin\left(-\frac{\pi n}{4}\right) = -\sin\left(\frac{\pi n}{4}\right)$$

Then from above we have:

$$y[n] = -2\sqrt{2}\sin\left(\frac{\pi(n+1)}{4}\right)$$

Problem 1.4 (OSB 2.55)

Yes. Suppose $x_1[n] = \cos(\omega n)$ and $x_2[n] = \cos((\omega + 2\pi)n)$. Then $x_1[n] = x_2[n]$ and the inputs are identical. All three systems behave deterministically, so the intermediate signals and the respective outputs A_1 and A_2 will be identical. Thus A will be periodic in ω (with period 2π). Recall that all distinct frequencies in discrete time fall within a continuous range of 2π. More generally for a sinusoidal input, the output of the overall system will be periodic in ω regardless of the systems in between the input and output (since the inputs $x_1[n]$ and $x_2[n]$ above will always be equal).

Problem 1.5 (OSB 3.4)

(a) If the Fourier transform is known to exist, then the ROC must include the unit circle. Thus, the ROC of $X(z)$ is $\frac{1}{3} < |z| < 2$ and therefore $x[n]$ is a two-sided sequence.

(b) Two: $\frac{1}{3} < |z| < 2$ and $2 < |z| < 3$.

(c) No. Stability requires the ROC to include the unit circle and causality requires the ROC to extend outward from the outermost pole and include $z = \infty$. It is not possible to satisfy both conditions at once, so it is not possible to have both stability and causality.

Problem 1.6 (OSB 3.9)

(a) For $H(z)$ to be causal the ROC must extend outward from the outermost pole and include $z = \infty$. The ROC is thus $|z| > \frac{1}{2}$.

(b) Yes, the system is stable since the ROC includes the unit circle.

(c)

$$y[n] = -\frac{1}{3}\left(-\frac{1}{4}\right)^n u[n] - \frac{4}{3}(2)^n u[-n-1]$$

$$Y(z) = -\frac{\frac{1}{3}}{1+\frac{1}{4}z^{-1}} + \frac{\frac{4}{3}}{1-2z^{-1}} = \frac{1+z^{-1}}{\left(1+\frac{1}{4}z^{-1}\right)\left(1-2z^{-1}\right)}$$

Since the first term of $y[n]$ is right-sided, the corresponding ROC constraint is $|z| > \frac{1}{4}$; likewise, the second term being left-sided leads to the ROC constraint $|z| < 2$. Therefore, the ROC for $Y(z)$ is $\frac{1}{4} < |z| < 2$.

$$X(z) = \frac{Y(z)}{H(z)} = \frac{\left(1+z^{-1}\right)}{\left(1+\frac{1}{4}z^{-1}\right)\left(1-2z^{-1}\right)} \frac{\left(1-\frac{1}{2}z^{-1}\right)\left(1+\frac{1}{4}z^{-1}\right)}{\left(1+z^{-1}\right)} = \frac{1-\frac{1}{2}z^{-1}}{1-2z^{-1}}$$

Only the pole at $z = 2$ remains, so the ROC for $X(z)$ is $|z| < 2$.

(d)

$$H(z) = \frac{1+z^{-1}}{\left(1-\frac{1}{2}z^{-1}\right)\left(1+\frac{1}{4}z^{-1}\right)} = \frac{2}{1-\frac{1}{2}z^{-1}} - \frac{1}{1+\frac{1}{4}z^{-1}}, \qquad |z| > \frac{1}{2}$$

$$h[n] = 2\left(\frac{1}{2}\right)^n u[n] - \left(-\frac{1}{4}\right)^n u[n]$$

(e)

$$H(z) = \frac{1+z^{-1}}{\left(1-\frac{1}{2}z^{-1}\right)\left(1+\frac{1}{4}z^{-1}\right)} = \frac{1+z^{-1}}{1-\frac{1}{4}z^{-1}-\frac{1}{8}z^{-2}} = \frac{Y(z)}{X(z)}$$

$$Y(z) - \frac{1}{4}z^{-1}Y(z) - \frac{1}{8}z^{-2}Y(z) = X(z) + z^{-1}X(z)$$

$$y[n] - \frac{1}{4}y[n-1] - \frac{1}{8}y[n-2] = x[n] + x[n-1].$$

Problem 1.7 (OSB 3.40)

(a) Translating the block diagram into z-transforms,

$$[X(z) - W(z)]\,H(z) + E(z) = W(z)$$

$$W(z) = \frac{H(z)}{1+H(z)}X(z) + \frac{1}{1+H(z)}E(z)$$

$$H_1(z) = \frac{H(z)}{1+H(z)} \qquad \text{and} \qquad H_2(z) = \frac{1}{1+H(z)}$$

(b) $H_1(z) = z^{-1}$

$H_2(z) = 1 - z^{-1}$

(c) $H(z)$ has a pole at $z = 1$ on the unit circle so it is not stable. But both $H_1(z)$ and $H_2(z)$ have poles at $z = 0$ and are stable ($H(z)$ is causal).

Problem 1.8 (OSB 3.46)

(a) The ROC must contain the unit circle in order for $y[n]$ to be stable, thus the ROC for $Y(z)$ is: $\frac{1}{2} < |z| < 2$.

(b) $y[n]$ is two-sided.

(c) Again the ROC must contain the unit circle for $x[n]$ to be stable, thus the ROC of $X(z)$ is: $|z| > \frac{3}{4}$.

(d) Yes, $x[n]$ is causal, since the ROC extends outward from the outermost pole and includes $z = \infty$.

(e) Since $x[n]$ is causal, we can use the initial-value theorem:

$$x[0] = \lim_{z\to\infty} X(z) = 0.$$

The limit goes to zero because $X(z)$ has a zero at $z = \infty$. Rational z-transforms have an equal number of poles and zeroes if we include the singularities at infinity. We can also verify the zero at $z = \infty$ by writing the mathematical expression for $X(z)$:

$$X(z) = \frac{Az^{-1}\left(1 - \frac{1}{4}z^{-1}\right)}{\left(1 + \frac{3}{4}z^{-1}\right)\left(1 - \frac{1}{2}z^{-1}\right)},$$

from which we see that as $z \to \infty, X(z) \to 0$.

(f) From the pole-zero diagrams for $Y(z)$ and $X(z)$ we have the following:

Poles of $Y(z)$: $z = \frac{1}{2}$ and $z = 2$. Zeroes of $Y(z)$: $z = 0$ and $z = \frac{1}{4}$.

Poles of $X(z)$: $z = -\frac{3}{4}$ and $z = \frac{1}{2}$. Zeroes of $X(z)$: $z = \infty$ and $z = \frac{1}{4}$.

$$H(z) = \frac{Y(z)}{X(z)}$$

Inverting $X(z)$ will turn the poles of $X(z)$ into zeroes and vice versa. As a result we have pole-zero cancellation at $z = \frac{1}{2}$ and $z = \frac{1}{4}$. $H(z)$ has zeroes at $z = 0$ and $z = -\frac{3}{4}$, and poles at $z = 2$ and $z = \infty$. Its ROC is $|z| < 2$.

Alternatively, we could have determined $H(z)$ from the mathematical expressions for $Y(z)$ and $X(z)$:

$$Y(z) = \frac{B\left(1 - \frac{1}{4}z^{-1}\right)}{\left(1 - \frac{1}{2}z^{-1}\right)\left(1 - 2z^{-1}\right)}$$

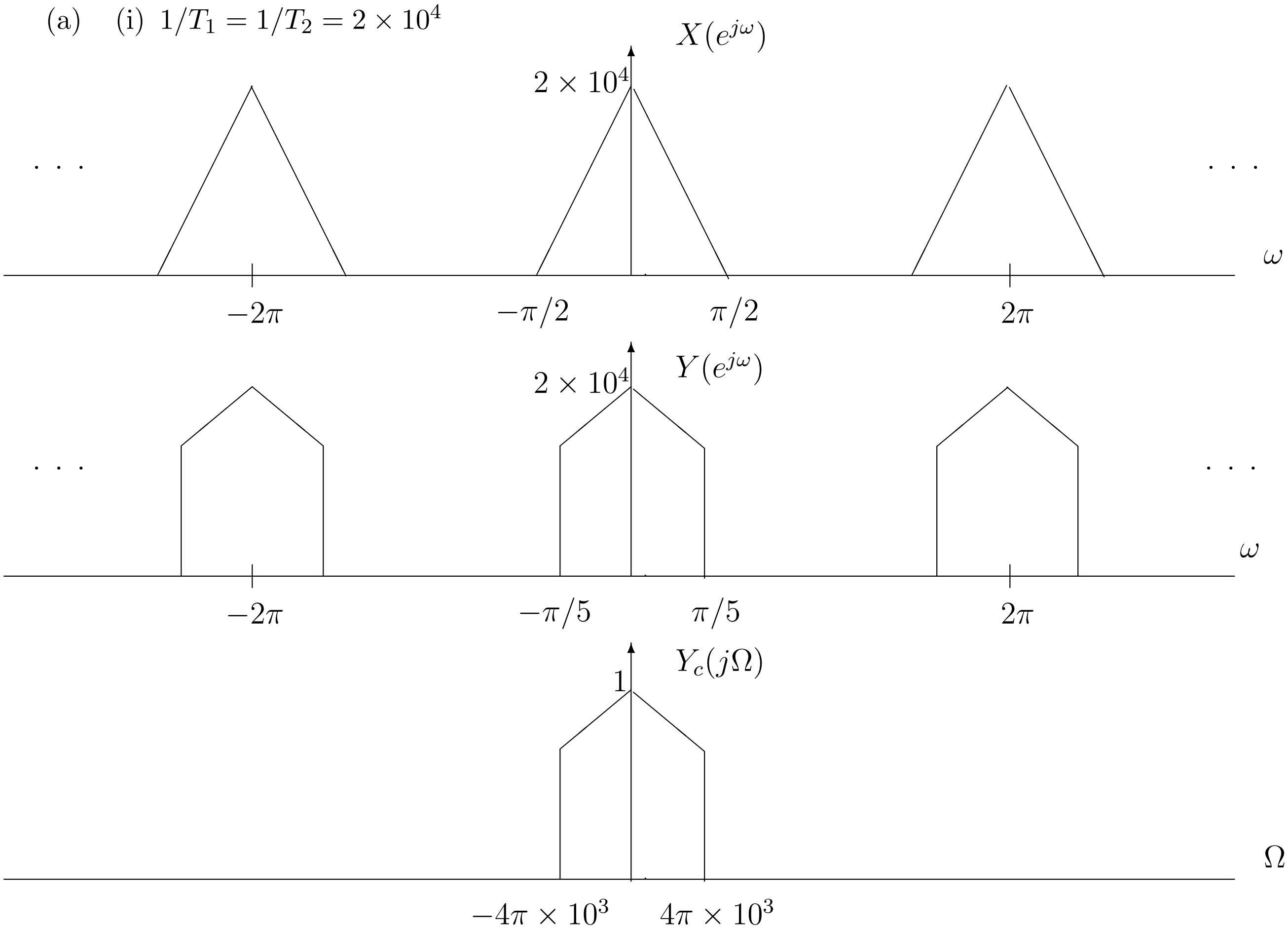
$$H(z) = \frac{Y(z)}{X(z)} = \frac{B\left(1 - \frac{1}{4}z^{-1}\right)}{\left(1 - \frac{1}{2}z^{-1}\right)\left(1 - 2z^{-1}\right)}\frac{\left(1 + \frac{3}{4}z^{-1}\right)\left(1 - \frac{1}{2}z^{-1}\right)}{Az^{-1}\left(1 - \frac{1}{4}z^{-1}\right)} = \frac{C\left(1 + \frac{3}{4}z^{-1}\right)}{z^{-1}\left(1 - 2z^{-1}\right)},$$

where $C = \frac{B}{A}$.

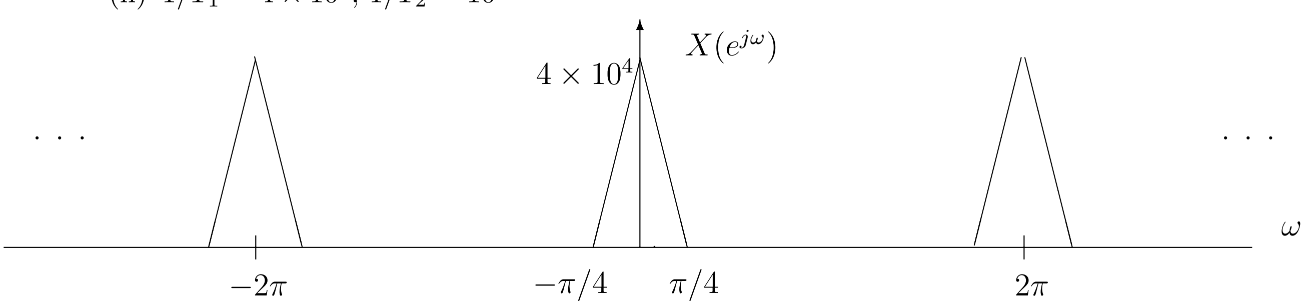
(g) Yes. $h[n]$ is anti-causal, since the ROC of $H(z)$ extends inward from the innermost pole and does include the origin ($z = 0$).

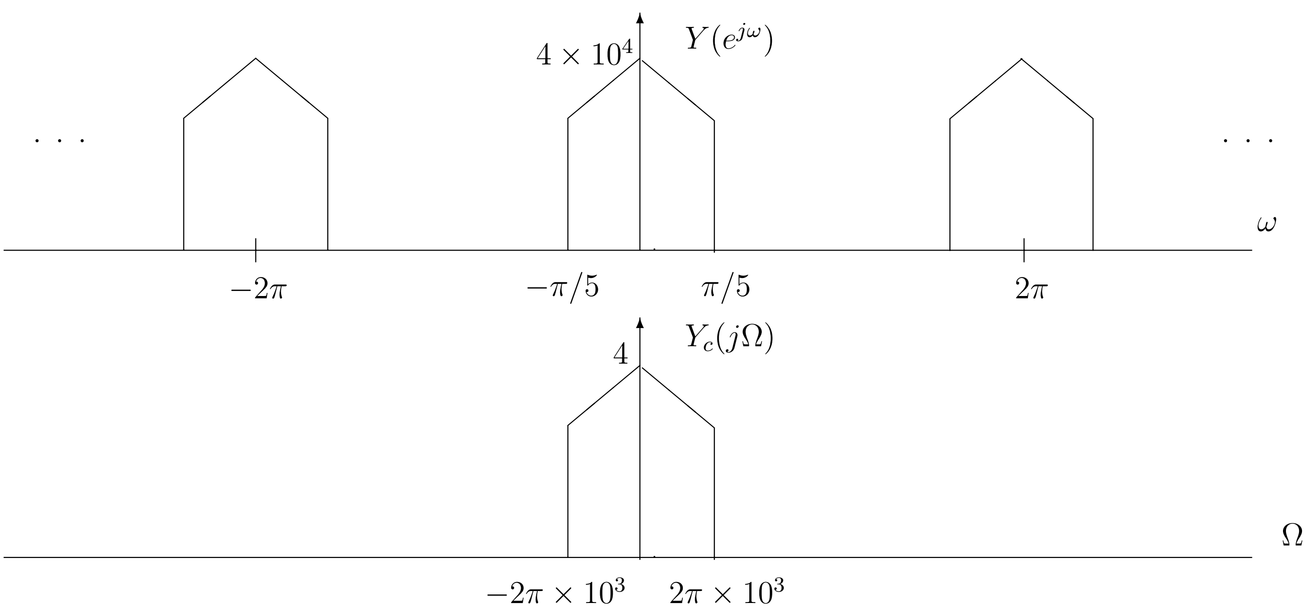
Problem 1.9

(a) (i) $1/T_1 = 1/T_2 = 2 \times 10^4$

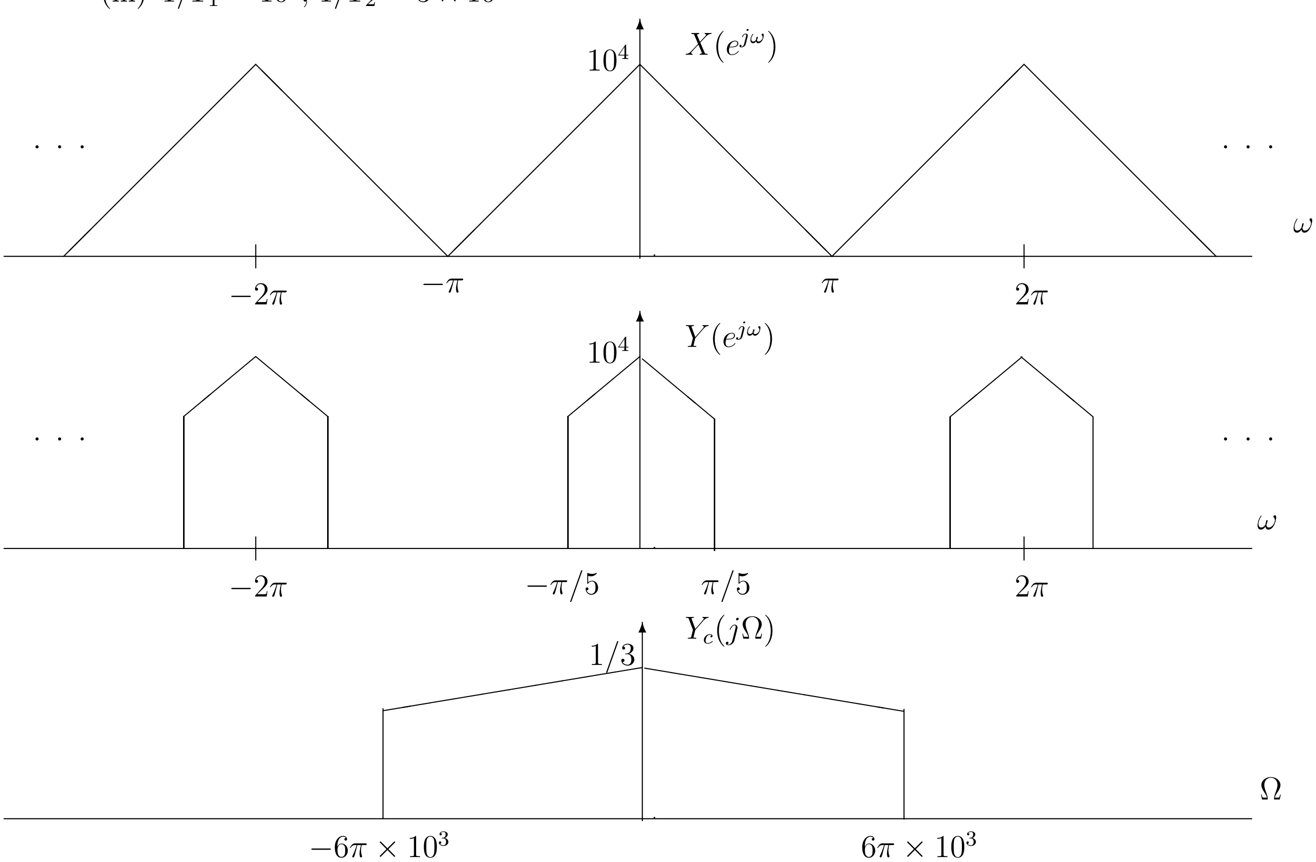


(ii) $1/T_1 = 4 \times 10^4$, $1/T_2 = 10^4$



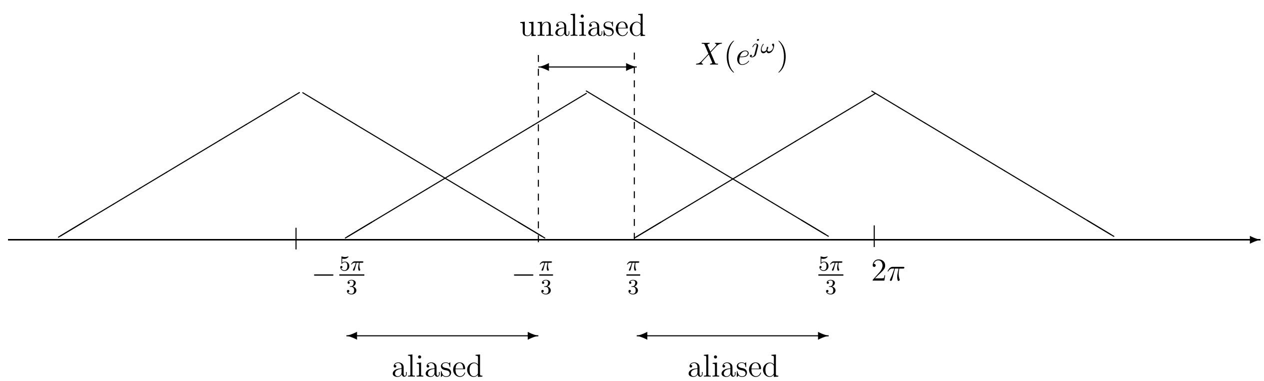


(iii) $1/T_1 = 10^4$, $1/T_2 = 3 \times 10^4$



(b) From the figure below, the only portion of the spectrum which remains unaffected by the aliasing is $|\omega| < \pi/3$. So if we choose $\omega_c < \pi/3$, the overall system is LTI with a frequency response of:

$$H_c(j\Omega) = \begin{cases} 1 & \text{for } |\Omega| < \omega_c \times 6 \times 10^3 \\ 0 & \text{otherwise.} \end{cases}$$



Problem 1.10

$y[n] = y_2[n]$

Justification:

The input signal $x[n]$ is made up of three narrow-band pulses: pulse-1 is a low-frequency pulse (whose peak is around 0.12π radians), pulse-2 is a higher-frequency pulse (0.3π radians), and pulse-3 is the highest-frequency pulse (0.5π radians).

Let $H(e^{j\omega})$ be the frequency response of Filter A. We read off the following values from the frequency response magnitude and group delay plots:

$$\begin{aligned} |H(e^{j(0.12\pi)})| &\approx 1.8 \\ |H(e^{j(0.3\pi)})| &\approx 1.7 \\ |H(e^{j(0.5\pi)})| &\approx 0 \\ \tau_g(0.12\pi) &\approx 40 \text{ samples} \\ \tau_g(0.3\pi) &\approx 80 \text{ samples} \end{aligned}$$

From these values, we would expect pulse-3 to be totally absent from the output signal $y[n]$. Pulse-1 will be scaled up by a factor of 1.8 and its envelope delayed by about 40 samples. Pulse-2 will be scaled up by a factor of 1.7 and its envelope delayed by about 80 samples. The correct output is thus $y_2[n]$.

Problem 1.11

Uniform Distribution: $f_x(x) = \begin{cases} \frac{1}{b-a} & a < x < b \\ 0 & \text{otherwise} \end{cases} = \begin{cases} \frac{1}{\Delta} & 0 < x < \Delta \\ 0 & \text{otherwise} \end{cases}$

$\mu = E[X] = \int\limits_{-\infty}^{\infty} x f_x(x) dx = \int\limits_{0}^{\Delta} \frac{1}{\Delta} x dx = \frac{\Delta}{2}$

$\sigma^2 = E[(X - E[X])^2] = E[X^2] - \mu^2 = \int\limits_{0}^{\Delta} \frac{1}{\Delta} x^2 dx - (\frac{\Delta}{2})^2 = \frac{\Delta^2}{3} - \frac{\Delta^2}{4} = \frac{\Delta^2}{12}$

Problem 1.12

(a) $R_{yx}[m] = R_{xx}[m] * h[m]$, but $R_{xy}[m] = R_{yx}[-m]$ and $R_{xx}[m] = R_{xx}[-m]$. Thus $R_{xy}[m] = R_{xx}[m] * h[-m]$. Since $R_{xx}[m] = \delta[m]$,

$$R_{xy}[m] = h[-m] = \begin{cases} 1 & m = -2, -1, 0 \\ 0 & \text{otherwise} \end{cases}$$

Then $R_{yy}[m] = R_{xx}[m] * h[m] * h[-m] = \begin{cases} 1 & m = -2, 2 \\ 2 & m = -1, 1 \\ 3 & m = 0 \\ 0 & \text{otherwise} \end{cases}$

(b)

$$\begin{aligned} P_{xx} &= 1 \\ P_{yy} &= 3 + 2e^{j\omega} + 2e^{-j\omega} + e^{2j\omega} + e^{-2j\omega} \\ &= 3 + 4\cos\omega + 2\cos(2\omega). \end{aligned}$$

Problem 1.13 (OSB 4.5)

Answers are in the back of the book.

Problem 1.14 (OSB 2.89)

(a)

$$\begin{aligned} E\{x[n]x[n]\} &= \phi_{xx}[0] \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \Phi_{xx}(e^{j\omega}) d\omega \end{aligned}$$

(b)

$$\begin{aligned}\Phi_{xx}(e^{j\omega}) &= \Phi_{ww}(e^{j\omega})|H(e^{j\omega})|^2 \\ &= \sigma_w^2 \frac{1}{1-\cos(\omega)+1/4} \\ &= \frac{\sigma_w^2}{5/4-\cos\omega}.\end{aligned}$$

(c)

$$\begin{aligned}\phi_{xx}[n] &= \phi_{ww}[n] * h[n] * h[-n] \\ &= \sigma_w^2 \left(\left(\frac{1}{2}\right)^n u[n] * \left(\frac{1}{2}\right)^{-n} u[-n] \right) \\ &= \frac{4}{3}\sigma_w^2 \left(\frac{1}{2}\right)^{|n|}.\end{aligned}$$