Massachusetts Institute of Technology
Department of Electrical Engineering and Computer Science

6.341: Discrete-Time Signal Processing

Fall 2005

Problem Set 11 Solutions

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Problem 11.1 (OSB 11.3)

Note: The answers in the back of the book may not be correct in your version of the textbook.

We factor $|X(e^{j\omega})|^2$ into:

$$\begin{aligned} |X(e^{j\omega})|^2 &= \frac{5}{4} - \cos\omega \\ &= \left(1 - \frac{1}{2}e^{-j\omega}\right)\left(1 - \frac{1}{2}e^{j\omega}\right) \\ &= X(e^{j\omega})X^*(e^{j\omega}) \end{aligned}$$

As a first attempt, we take

$$\begin{aligned} X(e^{j\omega}) &= 1 - \frac{1}{2}e^{-j\omega} \\ x[n] &= \delta[n] - \frac{1}{2}\delta[n-1] \end{aligned}$$

which does not satisfy the constraints $x[0] = 0$ and $x[1] > 0$.

We can modify the above choice by cascading it with an all-pass system, which will not affect the magnitude squared of the Fourier transform. Therefore we let

$$\begin{aligned} X(e^{j\omega}) &= \left(1 - \frac{1}{2}e^{-j\omega}\right)e^{-j\omega} \\ x[n] &= \delta[n-1] - \frac{1}{2}\delta[n-2] \end{aligned}$$

which does satisfy all of the constraints.

Another choice that works is to take the second factor in $|X(e^{j\omega})|^2$ and cascade it with $\left(-e^{-j2\omega}\right)$:

$$\begin{aligned} X(e^{j\omega}) &= \left(1 - \frac{1}{2}e^{j\omega}\right)\left(-e^{-j2\omega}\right) = \frac{1}{2}e^{-j\omega} - e^{-j2\omega} \\ x[n] &= \frac{1}{2}\delta[n-1] - \delta[n-2] \end{aligned}$$

Note that this second choice uses the zero at $z = 2$, the conjugate reciprocal of the zero at $z = \frac{1}{2}$ in the first choice. Conjugate reciprocal zeroes yield the same Fourier transform magnitude (up to a scaling).

Problem 11.2

The inverse DTFT of $j\text{Im}\{Y(e^{j\omega})\}$ is the odd part of $y[n]$, denoted by $y_o[n]$.

$$\begin{aligned} y_o[n] &= \text{DTFT}^{-1}[j3\sin\omega + j\sin 3\omega] \\ &= \text{DTFT}^{-1}\left[\frac{1}{2}\left(3e^{j\omega} - 3e^{-j\omega} + e^{j3\omega} - e^{-j3\omega}\right)\right] \\ &= \frac{1}{2}\left(3\delta[n+1] - 3\delta[n-1] + \delta[n+3] - \delta[n-3]\right) \end{aligned}$$

Since $y[n]$ is real and causal,

$$\begin{aligned} y[n] &= 2y_o[n]u[n] + y[0]\delta[n] \\ &= y[0]\delta[n] - 3\delta[n-1] - \delta[n-3] \end{aligned}$$

To determine $y[0]$, we use the fact that $Y(e^{j\omega})\Big|_{\omega=\pi} = 3$, i.e.,

$$\begin{aligned} Y(e^{j\omega})\Big|_{\omega=\pi} &= \sum_{n=-\infty}^{\infty} y[n](-1)^n \\ &= y[0] + 3 + 1 = 3 \\ y[0] &= -1 \end{aligned}$$

Therefore,

$$y[n] = -\delta[n] - 3\delta[n-1] - \delta[n-3]$$

Problem 11.3 (OSB 11.5)

In the frequency domain, the Hilbert transform is a 90° phase shifter:

$$H(e^{j\omega}) = \begin{cases} -j, & 0 < \omega < \pi \\ j, & -\pi < \omega < 0 \end{cases}$$

To find the Hilbert transform of each sequence, we will take the Fourier transform, multiply by $H(e^{j\omega})$, and take the inverse Fourier transform.

(a)

$$\begin{aligned} x_r[n] &= \cos\omega_0 n \\ X_r(e^{j\omega}) &= \pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0), \quad -\pi < \omega \le \pi \\ X_i(e^{j\omega}) &= H(e^{j\omega})X_r(e^{j\omega}) \\ &= -j\pi\delta(\omega - \omega_0) + j\pi\delta(\omega + \omega_0), \quad -\pi < \omega \le \pi \\ x_i[n] &= \sin\omega_0 n \end{aligned}$$

(b)

$$\begin{aligned}
x_r[n] &= \sin \omega_0 n \\
X_r(e^{j\omega}) &= \frac{\pi}{j}\delta(\omega - \omega_0) - \frac{\pi}{j}\delta(\omega + \omega_0), \quad -\pi < \omega \leq \pi \\
X_i(e^{j\omega}) &= -\pi\delta(\omega - \omega_0) - \pi\delta(\omega + \omega_0), \quad -\pi < \omega \leq \pi \\
x_i[n] &= -\cos \omega_0 n
\end{aligned}$$

(c) $x_r[n]$ is the impulse response of an ideal low-pass filter with cut-off frequency ω_c:

$$\begin{aligned}
x_r[n] &= \frac{\sin(\omega_c n)}{\pi n} \\
X_r(e^{j\omega}) &= \begin{cases} 1, & |\omega| < \omega_c \\ 0, & \omega_c < |\omega| \leq \pi \end{cases} \\
X_i(e^{j\omega}) &= \begin{cases} -j, & 0 < \omega < \omega_c \\ j, & -\omega_c < \omega < 0 \\ 0, & \omega_c < |\omega| \leq \pi \end{cases} \\
x_i[n] &= \frac{1}{2\pi}\int_{-\omega_c}^{0} j e^{j\omega n} d\omega - \frac{1}{2\pi}\int_{0}^{\omega_c} j e^{j\omega n} d\omega \\
&= \frac{1 - \cos(\omega_c n)}{\pi n}
\end{aligned}$$

Problem 11.4

The DTFT of $y_1[n]$ is given by

$$Y_1(e^{j\omega}) = X(e^{j\omega})e^{j\theta(\omega)}, \quad -\pi < \omega < \pi$$

The DTFT of $y_2[n]$ is given by

$$\begin{aligned}
Y_2(e^{j\omega}) &= \begin{cases} X(e^{j\omega})e^{j(\theta(\omega)-\pi/2)}, & 0 < \omega < \pi \\ X(e^{j\omega})e^{j(\theta(\omega)+\pi/2)}, & -\pi < \omega < 0 \end{cases} \\
&= \begin{cases} -jX(e^{j\omega})e^{j\theta(\omega)}, & 0 < \omega < \pi \\ jX(e^{j\omega})e^{j\theta(\omega)}, & -\pi < \omega < 0 \end{cases}
\end{aligned}$$

Since $w[n] = y_1[n] + jy_2[n]$,

$$\begin{aligned} W(e^{j\omega}) &= Y_1(e^{j\omega}) + jY_2(e^{j\omega}) \\ &= \begin{cases} X(e^{j\omega})e^{j\theta(\omega)}(1+1), & 0 < \omega < \pi \\ X(e^{j\omega})e^{j\theta(\omega)}(1-1), & -\pi < \omega < 0 \end{cases} \\ &= \begin{cases} 2X(e^{j\omega})e^{j\theta(\omega)}, & 0 < \omega < \pi \\ 0, & -\pi < \omega < 0 \end{cases} \end{aligned}$$

Therefore,

$$W(e^{j\omega}) = 0, \quad -\pi < \omega < 0$$

and since $|e^{j\theta(\omega)}| = 1$,

$$|W(e^{j\omega})| = 2|X(e^{j\omega})|, \quad 0 < \omega < \pi$$

Problem 11.5

We find the Fourier transform of $h[n]$ and then take its complex logarithm,

$$\begin{aligned} h[n] &= \delta[n] + \alpha\delta[n-n_0] \\ H(e^{j\omega}) &= 1 + \alpha e^{-j\omega n_0} \\ \hat{H}(e^{j\omega}) &= \log\left(1 + \alpha e^{-j\omega n_0}\right) \end{aligned}$$

The power series expansion for $\log(1+x)$ with $|x| < 1$ is given by:

$$\log(1+x) = \sum_{k=1}^{\infty}(-1)^{k+1}\frac{x^k}{k}$$

Letting $x = \alpha e^{-j\omega n_0}$ and checking that $|x| = |\alpha e^{-j\omega n_0}| = |\alpha| < 1$ as assumed, we obtain:

$$\hat{H}(e^{j\omega}) = \sum_{k=1}^{\infty}(-1)^{k+1}\frac{\alpha^k}{k}e^{-j\omega k n_0}$$

The complex cepstrum $\hat{h}[n]$ is found by taking the inverse Fourier transform of $\hat{H}(e^{j\omega})$ and identifying $e^{-j\omega k n_0} \leftrightarrow \delta[n-kn_0]$:

$$\hat{h}[n] = \sum_{k=1}^{\infty}(-1)^{k+1}\frac{\alpha^k}{k}\delta[n-kn_0]$$

$\hat{h}[n]$ is plotted in Figure 11.5-1:

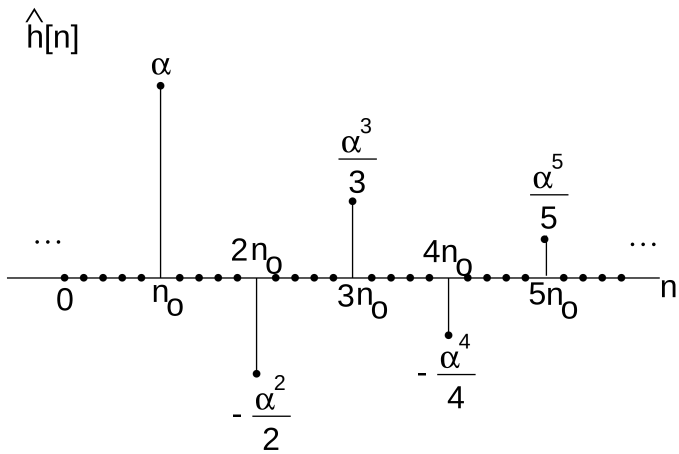

Figure 11.5-1: Complex cepstrum $\hat{h}[n]$ for an echo system.

Problem 11.6

The sequence $x[n]$ being minimum-phase means that $x[n]$ is also causal, so that $x[n] = 0,\ n < 0$. As stated in the problem, minimum phase implies that the complex cepstrum $\hat{x}[n]$ is causal, i.e. $\hat{x}[n] = 0,\ n < 0$. Thus the lower bound on the sum in equation (12.34) becomes $k = 0$ (because of $\hat{x}[k]$), while the upper bound becomes $k = n$ (because of $x[n-k]$).

$$x[n] = \sum_{k=0}^{n} \left(\frac{k}{n}\right) \hat{x}[k]x[n-k], \quad n > 0$$

Isolating the $k = n$ term from the sum and solving for $\hat{x}[n]$,

$$x[n] = \hat{x}[n]x[0] + \sum_{k=0}^{n-1} \left(\frac{k}{n}\right) \hat{x}[k]x[n-k]$$

$$\hat{x}[n] = \frac{x[n]}{x[0]} - \sum_{k=0}^{n-1} \left(\frac{k}{n}\right) \hat{x}[k]\frac{x[n-k]}{x[0]}, \quad n > 0$$

The equation above is a recursion formula for $\hat{x}[n],\ n > 0$, while we know that $\hat{x}[n] = 0$ for $n < 0$:

$$\hat{x}[n] = \begin{cases} 0, & n < 0 \\ \dfrac{x[n]}{x[0]} - \displaystyle\sum_{k=0}^{n-1} \left(\frac{k}{n}\right) \hat{x}[k]\frac{x[n-k]}{x[0]}, & n > 0 \end{cases}$$

However, the recursion cannot determine $\hat{x}[0]$, so we must find it through some other means. Recall the initial-value theorem for a causal sequence $x[n]$ such that $x[n] = 0$ for $n < 0$:

$$x[0] = \lim_{z \to \infty} X(z)$$

Since $\hat{x}[n]$ is also zero for $n < 0$,

$$\hat{x}[0] = \lim_{z \to \infty} \hat{X}(z)$$

But $\hat{X}(z) = \log X(z)$, so we have

$$\begin{aligned}\hat{x}[0] &= \lim_{z \to \infty} \log X(z) \\ &= \log \left(\lim_{z \to \infty} X(z) \right) \\ &= \log (x[0])\end{aligned}$$

Thus we can determine $\hat{x}[0]$ using only $x[0]$, so the computation is causal.

Now suppose that $\hat{x}[n]$ is known for $0 \leq n \leq n_0 - 1$. Using the recursion, we are able to calculate the next value $\hat{x}[n_0]$ from the known past values of $\hat{x}[n]$ and from the values of $x[n]$ for $0 \leq n \leq n_0$. To start the recursion, we determine $\hat{x}[0]$, which is then used to determine $\hat{x}[1]$, and so on. $\hat{x}[n]$ can therefore be recursively computed. Furthermore, the computation of $\hat{x}[n_0]$ for any $n_0 \geq 0$ only involves values of $x[n]$ for $0 \leq n \leq n_0$, so the recursion can be implemented in a causal manner.

Problem 11.7

(a) Similar to Problem 11.2, the inverse DTFT of $\mathrm{Re}\{X(e^{j\omega})\}$ is the even part $x_e[n]$ of $x[n]$.

$$\begin{aligned}x_e[n] &= \mathrm{DTFT}^{-1}[1 + 3\cos\omega + \cos 3\omega] \\ &= \delta[n] + \frac{1}{2}\left(3\delta[n+1] + 3\delta[n-1] + \delta[n+3] + \delta[n-3]\right)\end{aligned}$$

Since $x[n]$ is real and causal, it can be *uniquely* determined from its even part $x_e[n]$:

$$\begin{aligned}x[n] &= 2x_e[n]u[n] - x_e[0]\delta[n] \\ x[n] &= \delta[n] + 3\delta[n-1] + \delta[n-3]\end{aligned}$$

(b) Let $X(e^{j\omega})$ and $\hat{X}(e^{j\omega})$ denote the Fourier transforms of the sequence $x[n]$ and its complex cepstrum $\hat{x}[n]$. $X(e^{j\omega})$ and $\hat{X}(e^{j\omega})$ are related by:

$$\hat{X}(e^{j\omega}) = \log|X(e^{j\omega})| + j \arg\left[X(e^{j\omega})\right]$$

where $\arg\left[X(e^{j\omega})\right]$ denotes the continuous unwrapped phase.

If $x_1[n] = x[-n]$, then $X_1(e^{j\omega}) = X(e^{-j\omega})$, and

$$\begin{aligned}\hat{X}_1(e^{j\omega}) &= \log|X_1(e^{j\omega})| + j \arg\left[X_1(e^{j\omega})\right] \\ &= \log|X(e^{-j\omega})| + j \arg\left[X(e^{-j\omega})\right] \\ &= \hat{X}(e^{-j\omega})\end{aligned}$$

Therefore $\hat{x}_1[n] = \hat{x}[-n]$ also. Statement 1 is **true**.

If $x[n]$ is real, then the Fourier transform magnitude $|X(e^{j\omega})|$ is an even function of ω, while the unwrapped phase is an odd function of ω. The real part of $\hat{X}(e^{j\omega})$, which is the logarithm of $|X(e^{j\omega})|$, must be an even function, while the imaginary part of $\hat{X}(e^{j\omega})$ is equal to $\arg\left[X(e^{j\omega})\right]$ and must be an odd function. Therefore $\hat{X}(e^{j\omega})$ is conjugate symmetric and the complex cepstrum $\hat{x}[n]$ is real.

Statement 2 is also **true**.