Massachusetts Institute of Technology
Department of Electrical Engineering and Computer Science

6.341: Discrete-Time Signal Processing

Fall 2005

Solutions for Problem Set 10

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Problem 10.1

Problem 1, Fall 2004 Final Exam

We begin by finding an expression for $G[k]$:

$$G[k] = \sum_{n=-\infty}^{\infty} g[n]e^{-j2\pi nk/N} = \sum_{n=-\infty}^{\infty} x[n]w[n]e^{-j2\pi nk/N} \tag{1}$$

Since we're given that $h_k[n] = e^{j2\pi nk/N}$, we can solve for $v_k[m]$:

$$v_k[m] = \sum_{n=-\infty}^{\infty} x[m]h_k[m-n] = \sum_{n=-\infty}^{\infty} x[m]h_0[m-n]e^{j2\pi(m-n)k/N} \tag{2}$$

Evaluating (2) for $m = 0$ gives

$$v_k[0] = \sum_{n=-\infty}^{\infty} x[n]h_0[-n]e^{-j2\pi nk/N} = G[k] \ , \tag{3}$$

and comparing (1) and (3) shows us that $w[n]$ and $h_0[n]$ are related by

$$w[n] = h_0[-n] = \begin{cases} 0.9^{-n}, & -M+1 \le n \le 0 \\ 0, & \text{otherwise} \end{cases} \ .$$

Problem 10.2

$X_w[k]$ is defined as

$$X_w[k] = \sum_{n=0}^{255} x[n]0.9^{-n}e^{-j\frac{2\pi}{256}kn},$$

which is what we'd like our system to eventually implement. In terms of $v_k[n]$ this is

$$X_w[k] = v_k[N_k] = \sum_{n=-\infty}^{\infty} x[n]h_k[N_k - n] = \sum_{n=-\infty}^{\infty} x[n]h_0[N_k - n]e^{-j\omega_k(N_k-n)}.$$

We can allow the limits of the sum to go from $n = 0$ to $n = 255$ if we restrict $h_0[N_k - n]$ to be possibly nonzero only for $N_k - n \geq 0$ and $N_k - n \leq 255$, or equivalently, for $N_k - 255 \leq n \leq N_k$. Since the prototype filter must be causal, $N_k - 255$ (the lower limit on the filter's possibly nonzero region) must be greater than or equal to 0. N_k can then be judiciously chosen to be

$$N_k = 256 \quad \forall k.$$

We now have

$$v_k[256] = \sum_{n=0}^{255} x[n]h_0[256 - n]e^{-j\omega_k(256-n)}.$$

Putting issues with the exponential term aside for the moment, we know we'd like to have

$$h_0[256 - n] = \begin{cases} 0.9^{-n}, & 0 \leq n \leq 255 \\ 0, & \text{otherwise} \end{cases}.$$

With a change of variables this becomes

$$h_0[n] = \begin{cases} 0.9^{n-256}, & 1 \leq n \leq 256 \\ 0, & \text{otherwise} \end{cases},$$

and so we now have

$$v_k[256] = \sum_{n=0}^{255} x[n]0.9^{-n}e^{-j\omega_k(256-n)}.$$

We'd still like

$$e^{-j\omega_k(256-n)} = e^{-j\frac{2\pi}{256}kn},$$

which is satisfied for

$$\omega_k = -\frac{2\pi}{256}k.$$

We now have

$$v_k[256] = \sum_{n=0}^{255} x[n]0.9^{-n}e^{j\frac{2\pi}{256}k(256-n)} = \sum_{n=0}^{255} x[n]0.9^{-n}e^{j2\pi k}e^{-j\frac{2\pi}{256}kn} = \sum_{n=0}^{255} x[n]0.9^{-n}e^{-j\frac{2\pi}{256}kn}.$$

Problem 10.3

(a) $L = 256$ and $R = 1$

(b) M: (a), ω_k: (b), a_l: (a)

Problem 10.4

OSB Problem 10.40, (a) - (d)

(a)

$$\begin{aligned} X[n,\lambda) &= \sum_{m=-\infty}^{\infty} x[n+m]w[m]e^{-j\lambda m} \\ &= \sum_{m'=-\infty}^{\infty} x[m']w[m'-n]e^{-j\lambda m'}e^{j\lambda n} \\ &\overset{h_0[n]=w[-n]}{=} e^{j\lambda n} \sum_{m'=-\infty}^{\infty} (x[m']e^{-j\lambda m'})h_0[n-m'] \\ &\overset{x'[n]=x[n]e^{-j\lambda n}}{=} e^{j\lambda n} x'[n] * h_0[n] \end{aligned}$$

Now we show that $X[n,\lambda)$ is the output of the system of Figure P10.40-1 if $h_0[n] = w[-n]$ holds.

Obviously it is LTI since $e^{-j\lambda n}$ can be treated as constant when λ is fixed.

When $x[n] = \delta[n]$, the input to filter $h_0[n]$ is still $\delta[n]$. The output of filter $h_0[n]$ is $h_0[n] = w[-n]$. Thus, the impulse response of the equivalent LTI system is:

$$h_{eq}[n] = w[-n]e^{j\lambda n}.$$

The frequency response of the equivalent LTI system is

$$H_{eq}(e^{j\omega}) = W(e^{j(\lambda-\omega)}).$$

(b) Similar to part(a), when $x[n] = \delta[n]$,

$$\begin{aligned} s[n] &= h_o[n] = w[-n] \\ S(e^{j\omega}) &= W(e^{-j\omega}) \end{aligned}$$

For typical window sequences $w[n]$, $W(e^{j\omega})$ has a lowpass discrete-time Fourier transform. Therefore, $S(e^{j\omega}) = W(e^{j(-\omega)})$ should also have a lowpass discrete-time Fourier transform, while $H_{eq}(e^{j\omega}) = W(e^{j(\lambda-\omega)})$ have a bandpass discrete-time Fourier transform.

(c) Based on conclusion from part (a),

we have:

$$
\begin{aligned}
y_0[n] &= X[n,\lambda_0) \\
y_1[n] &= X[n,\lambda_1) \\
&\cdots \\
y_i[n] &= X[n,\lambda_i) \\
&\cdots \\
y_{N-1}[n] &= X[n,\lambda_{N-1})
\end{aligned}
$$

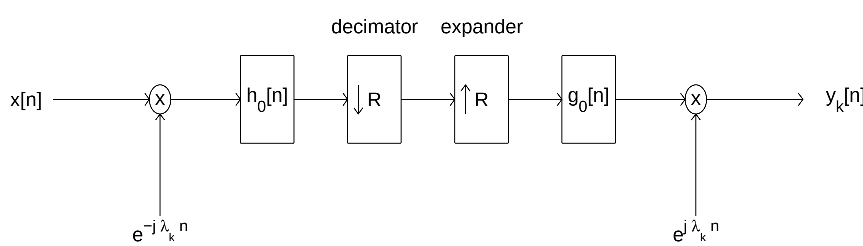
In total,

$$
\begin{aligned}
y[n] &= \sum_{i=0}^{N-1} X[n,\lambda_i) \\
&= \sum_{i=0}^{N-1} \sum_{m=-\infty}^{\infty} x[n+m]w[m]e^{-j\lambda_i m} \\
&= \sum_{m=-\infty}^{\infty} x[n+m]w[m] \sum_{i=0}^{N-1} e^{-j\lambda_i m}
\end{aligned}
$$

Since we assume $N \geq L \geq R$, we can consider only the items when $|m| \leq N$. Thus,

$$
\begin{aligned}
y[n] &= \sum_{m=-\infty}^{\infty} x[n+m]w[m]N\delta[m] \\
&= Nx[n]w[0]
\end{aligned}
$$

(d) Consider a single channel,



In the frequency domain, the input to the decimator is

$$X\left(e^{j(\omega+\lambda_k)}\right) H_0(e^{j\omega})$$

so the output of the decimator is

$$\frac{1}{R}\sum_{l=0}^{R-1} X\left(e^{j((\omega-2\pi l)/R+\lambda_k)}\right) H_0\left(e^{j(\omega-2\pi l)/R}\right)$$

The output of the expander is

$$\frac{1}{R}\sum_{l=0}^{R-1} X\left(e^{j(\omega+\lambda_k-2\pi l/R)}\right) H_0\left(e^{j(\omega-2\pi l/R)}\right)$$

The output $Y_k(e^{j\omega})$ is then

$$Y_k(e^{j\omega}) = \frac{1}{R}\sum_{l=0}^{R-1} G_0\left(e^{j(\omega-\lambda_k)}\right) X\left(e^{j(\omega-2\pi l/R)}\right) H_0\left(e^{j(\omega-\lambda_k-2\pi l/R)}\right)$$

The overall system output is formed by summing these terms over k.

$$\begin{aligned} Y(e^{j\omega}) &= \sum_{k=0}^{N-1} Y_k(e^{j\omega}) \\ &= \frac{1}{R}\sum_{l=0}^{R-1}\sum_{k=0}^{N-1} G_0\left(e^{j(\omega-\lambda_k)}\right) X\left(e^{j(\omega-2\pi l/R)}\right) H_0\left(e^{j(\omega-\lambda_k-2\pi l/R)}\right) \end{aligned}$$

To cancel the aliasing, we rewrite the equation as follows:

$$\begin{aligned} Y(e^{j\omega}) &= X(e^{j\omega})\frac{1}{R}\sum_{k=0}^{N-1} H_0\left(e^{j(\omega-\lambda_k)}\right) G_0\left(e^{j(\omega-\lambda_k)}\right) \\ &+ \underbrace{\sum_{l=1}^{R-1} X\left(e^{j(\omega-2\pi l/R)}\right)\frac{1}{R}\sum_{k=0}^{N-1} G_0\left(e^{j(\omega-\lambda_k)}\right) H_0\left(e^{j(\omega-\lambda_k-2\pi l/R)}\right)}_{\text{Aliasing Component}} \end{aligned}$$

Therefore, we require the following relations to be satisfied so that $y[n] = x[n]$:

$$\begin{aligned} \sum_{k=0}^{N-1} G_0\left(e^{j(\omega-\lambda_k)}\right) H_0\left(e^{j(\omega-\lambda_k-2\pi l/R)}\right) &= 0, \qquad \forall\ \omega, \text{ and } l = 1,\ldots,R-1 \\ \sum_{k=0}^{N-1} H_0\left(e^{j(\omega-\lambda_k)}\right) G_0\left(e^{j(\omega-\lambda_k)}\right) &= R, \qquad \forall\ \omega \end{aligned}$$