# Lectures 8 & 9

# M/G/1 Queues

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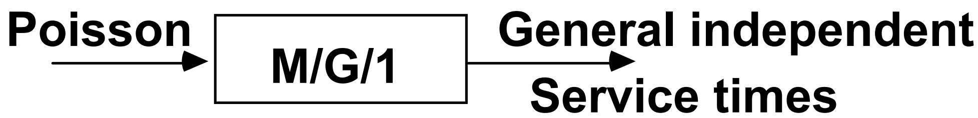
# M/G/1 QUEUE




- Poisson arrivals at rate $\lambda$
- Service time has arbitrary distribution with given E[X] and $E[X^2]$
  – Service times are independent and identically distributed (IID)
  – Independent of arrival times
  – E[service time] = $1/\mu$
  – Single Server queue

# Pollaczek-Khinchin (P-K) Formula

$$W = \frac{\lambda E[X^2]}{2(1-\rho)}$$

**where $\rho = \lambda/\mu = \lambda E[X]$ = line utilization**

**From Little's formula,**

$$N_Q = \lambda W$$

$$T = E[X] + W$$

$$N = \lambda T = N_Q + \rho$$

# M/G/1 EXAMPLES

- **Example 1: M/M/1**

  **$E[X] = 1/\mu$ ; $E[X^2] = 2/\mu^2$**

$$W = \frac{\lambda}{\mu^2(1-\rho)} = \frac{\rho}{\mu(1-\rho)}$$

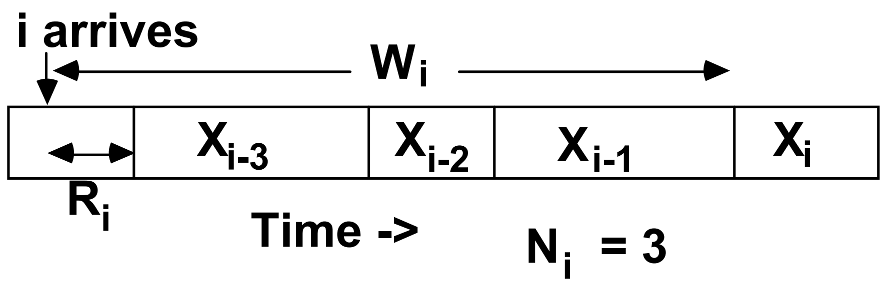
- **Example 2: M/D/1 (Constant service time $1/\mu$)**

  **$E[X] = 1/\mu$ ; $E[X^2] = 1/\mu^2$**

$$W = \frac{\lambda}{2\mu^2(1-\rho)} = \frac{\rho}{2\mu(1-\rho)}$$

# Proof of Pollaczek-Khinchin

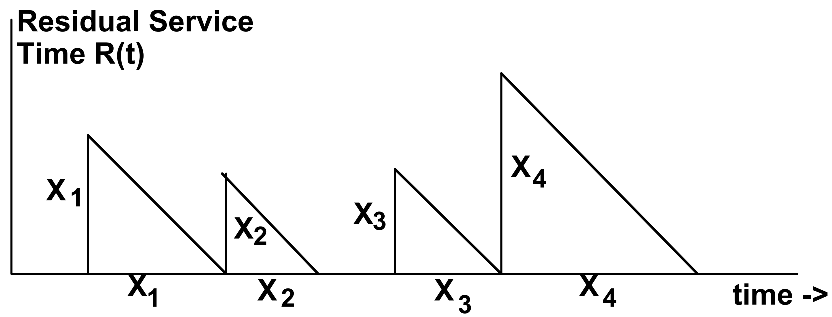
- Let $W_i$ = waiting time in queue of i[th] arrival
  $R_i$ = Residual service time seen by I (I.e., amount of time for current customer receiving service to be done)
  $N_i$ = Number of customers found in queue by i




$$W_i = R_i + \sum_{j=i-N_i}^{i-1} X_j$$

- $E[W_i] = E[R_i] + E[X]E[N_i] = R + N_Q/\mu$
  - Here we have used PASTA property plus independent service time property
- $W = R + \lambda W/\mu \Rightarrow W = R/(1-\rho)$
  - Using little's formula

# What is R?
# (Time Average Residual Service Time)




**Let M(t) = Number of customers served by time t**

**E[R(t)] = 1/t (sum of area in triangles)**

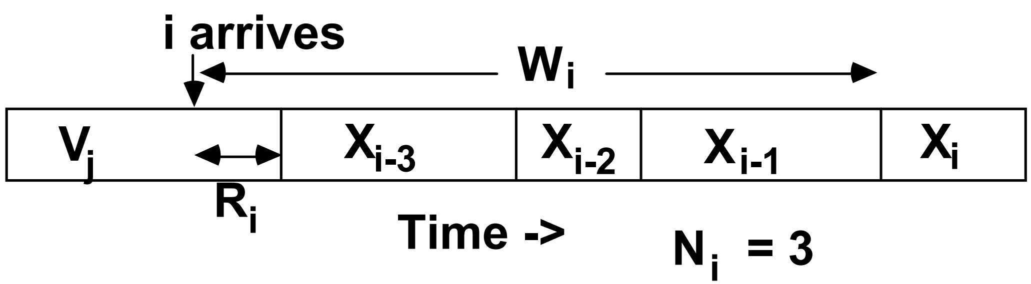
$$\mathbf{R}_t = \frac{1}{t}\int_0^t R(\tau)\,d\tau = \frac{1}{t}\sum_{i=1}^{M(t)} X_i^2 = \frac{1}{2}\,\frac{M(t)}{t}\sum_{i=1}^{M(t)}\frac{X_i^2}{M(t)}$$

**As t -> Infinity** $\frac{M(t)}{t}$ **= average departure rate = average arrival rate**

$$\frac{M(t)}{t}\sum_{i=1}^{M(t)}\frac{X_i^2}{M(t)} = E[X^2] \quad \Rightarrow R = \lambda E[X^2]/2$$

# M/G/1 Queue with Vacations

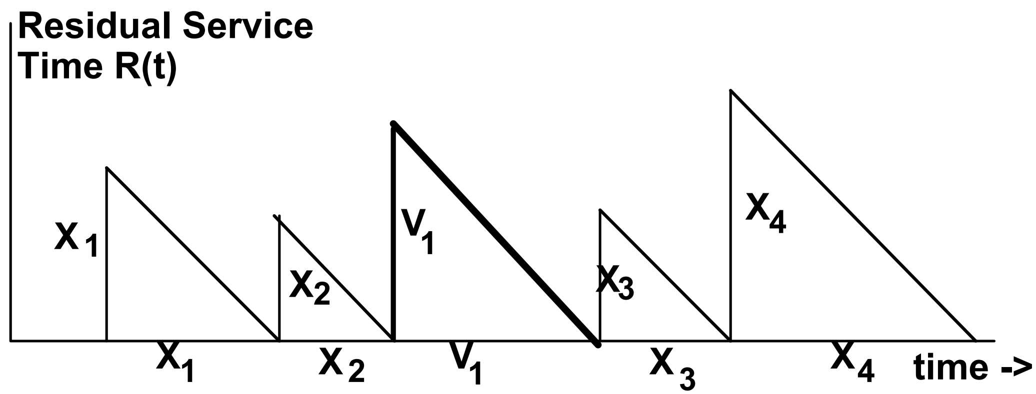
- **Useful for polling and reservation systems (e.g., token rings)**
- **When the queue is empty, the server takes a vacation**
- **Vacation times are IID and independent of service times and arrival times**
  - **If system is empty after a vacation, the server takes another vacation**
  - **The only impact on the analysis is that a packet arriving to an empty system must wait for the end of the vacation**




$$W_i = R_i + \sum_{j=i-N_i}^{i-1} X_j$$

$$E[W_i] = E[R_i] + E[X]E[N_i] = R + N_Q/\mu = R/(1-\rho)$$

# Average Residual Service Time (with vacations)




$$R = [R(t)] = \frac{1}{t}\int_0^t R(\tau)\,d\tau = \frac{1}{t}\left(\sum_{i=1}^{M(t)} \frac{X_i^2}{2} + \sum_{j=1}^{L(t)} \frac{V_j^2}{2}\right)$$

$$R = \lim_{t \to \infty} \frac{E[M(t)]}{t}\frac{E[X^2]}{2} + \frac{L(t)}{t}\frac{E[V^2]}{2}$$

- **Where L(t) is the number of vacations taken up to time t**
- **M(t) is the number of customers served by time t**

# Average Residual Service Time (with vacations)

- **As $t \to \infty$, $M(t)/t \to \lambda$ and $L(t)/t \to \lambda_v$ = vacation rate**

- **Now, let I = 1 if system is on vacation and I = 0 if system is busy**
- **By Little's Theorem we have,**
  - **E[I] =E[#vacations] = P(system idle) = $1-\rho = \lambda_v E[V]$**
  - **=> $\lambda_v = (1-\rho)/E[V]$**

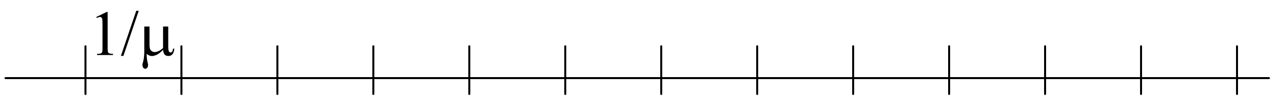
- **Hence,**

**remember $W = R/(1-\rho)$**

$$R = \lambda \frac{E[X^2]}{2} + \frac{(1-\rho)E[V^2]}{2E[V]}$$

$$W = \lambda \frac{E[X^2]}{2(1-\rho)} + \frac{E[V^2]}{2E[V]}$$

# Example: Slotted M/D/1 system




**Each slot = one packet transmission time = $1/\mu$**

- **Transmission can begin only at start of a slot**
- **If system is empty at the start of a slot, server not available for the duration of the slot (vacation)**

- **E[X] = E[v] =** $1/\mu$
- **E[X²] = E[v²] =** $1/\mu^2$

$$W = \frac{\lambda/\mu^2}{2(1-\lambda/\mu)} + \frac{1/\mu^2}{2/\mu} = \frac{\lambda/\mu}{2(\mu-\lambda)} + \frac{1/\mu}{2}$$

$$= W_{M/D/1} + E[X]/2$$

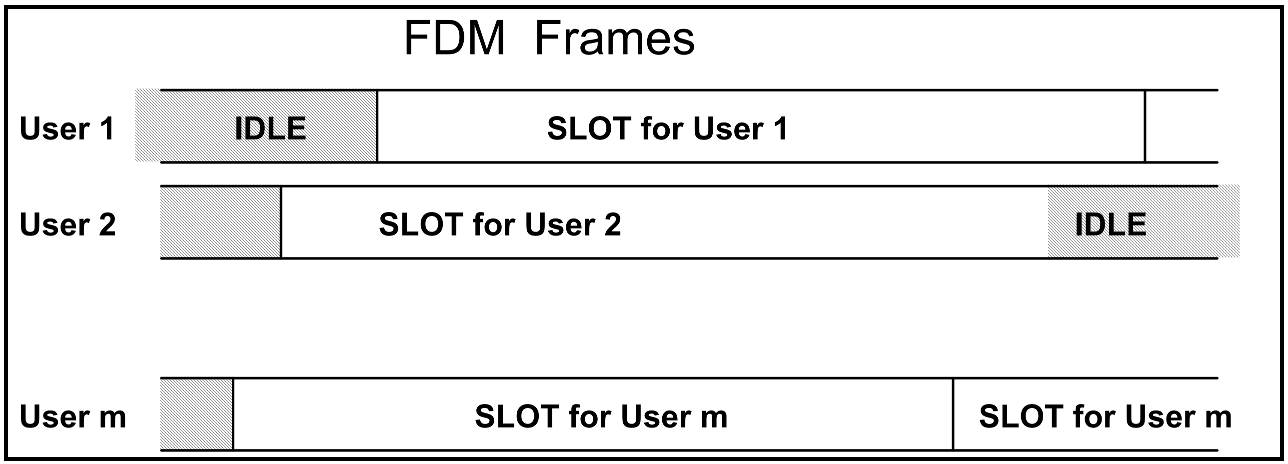
- **Notice that an average of 1/2 slot is spent waiting for the start of a slot**

# FDM EXAMPLE

- **Assume m Poisson streams of fixed length packets of arrival rate $\lambda/m$ each multiplexed by FDM on m subchannels. Total traffic = $\lambda$**

  **Suppose it takes m time units to transmit a packet, so $\mu=1/m$.**

  **The total system load: $\rho = \lambda$**

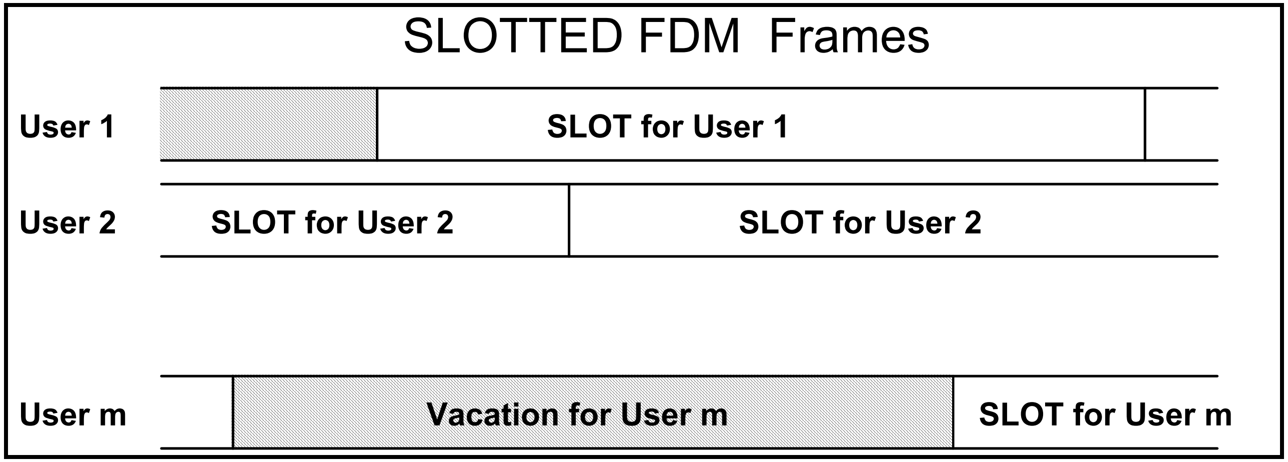
- 


- **We have an M/D/1 system { $W=\lambda E[x2]/2(1-\rho)$ }**

$$W_{FDM} = \frac{(\lambda/m)\, m^2}{2\,(1-\rho)} = \frac{\rho\ \ m}{2\,(1-\rho)}$$

# Slotted FDM

- **Suppose now that system is slotted and transmissions start only on m time unit boundaries.**



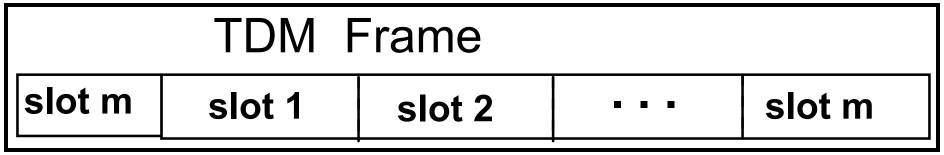

- **This is M/D/1 with <u>vacations</u>**
  - **Server goes on vacation for m time units when there is nothing to transmit**

    $E[V] = m;\ E[V^2] = m^2.$

$$W_{SFDM} = W_{FDM} + E[V^2]/2E[V]$$

$$= W_{FDM} + m/2$$

# TDM EXAMPLE




- **TDM with one packet slots is the same (a session has to wait for its own slot boundary), so**

**$W = R/(1-\rho)$**

$$R = \lambda \frac{E[X^2]}{2} + \frac{(1-\rho)E[V^2]}{2E[V]}$$

$$W = \lambda \frac{E[X^2]}{2(1-\rho)} + \frac{E[V^2]}{2E[V]}$$

# TDM EXAMPLE

- **Therefore,** $W_{TDM} = W_{FDM} + m/2$

  **Adding the packet transmission time, TDM comes out best because transmission time = 1 instead of m.**

$$T_{FDM} = [W_{FDM}] + m$$

$$T_{SFDM} = [W_{FDM} + m/2] + m$$

$$T_{TDM} = [W_{FDM} + m/2] + 1$$

$$= T_{FDM} - [m/2 - 1]$$