6.262: Discrete Stochastic Processes 3/30/11

Reminder: Quiz, 4/4/11, 7 - 9:30pm, Room 32-141
Sections of notes not covered: 1.5.3-4, 2.4, 3.5.3, 3.6, 4.6-8

For text with most errors corrected, see
http://www.rle.mit.edu/rgallager/notes.htm

Lecture 15: The last(?) renewal

Outline:

- Review sample-path averages and Wald
- Little's theorem
- Markov chains and renewal processes
- Expected number of renewals, $m(t) = \mathsf{E}[N(t)]$
- Elementary renewal and Blackwell thms
- Delayed renewal processes

One of the main reasons why the concept of convergence WP1 is so important is the following:

Thm: Assume that $\{Z_n;\, n \geq 1\}$ converges to α WP1 and assume that $f(x)$ is a real valued function of a real variable that is continuous at $x = \alpha$. Then $\{f(Z_n);\, n \geq 1\}$ converges WP1 to $f(\alpha)$.

For a renewal process with interarrivals $\{X_n; n \geq 1\}$ where $\mathsf{E}[X] < \infty$, the arrival epochs satisfy $S_n/n \to \mathsf{E}[X]$ WP1 and thus $n/S_n \to 1/\overline{X}$ WP1. The strong law for renewals follows.

Thm: $\Pr\Big\{\lim_{t\to\infty} \frac{N(t)}{t} = \frac{1}{\overline{X}}\Big\} = 1$.

The strong law for renewals also holds if $\overline{X} = \infty$. In this case, since X is a rv and S_n is a rv for all n, $N(t)$ grows without bound as $t \to \infty$, but $N(t)/t \to 0$.

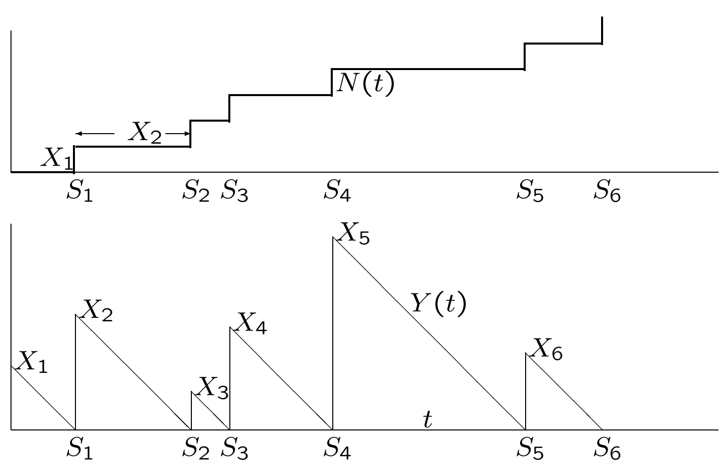
Since $N(t)/t$ converges WP1 to $1/\overline{X}$, it also must converge in probability, i.e.,

$$\lim_{t\to\infty} \Pr\left\{\left|\frac{N(t)}{t} - \frac{1}{\overline{X}}\right| > \epsilon\right\} = 0 \qquad \text{for all } \epsilon > 0$$

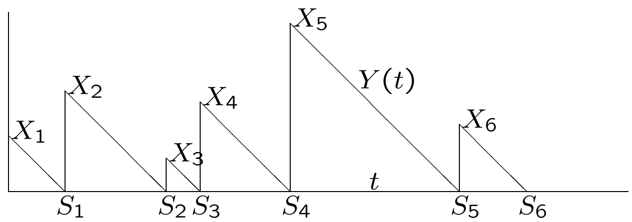
This is similar to the elementary renewal theorem, which says that

$$\lim_{t\to\infty} \mathsf{E}\left[\frac{N(t)}{t}\right] = \frac{1}{\overline{X}}$$

Residual life



The integral of $Y(t)$ over t is a sum of terms $X_n^2/2$.



$$\frac{1}{2t}\sum_{n=1}^{N(t)} X_n^2 \le \frac{1}{t}\int_0^t Y(\tau)d\tau \le \frac{1}{2t}\sum_{n=1}^{N(t)+1} X_n^2$$

$$\lim_{t\to\infty} \frac{\sum_{n=1}^{N(t)} X_n^2}{2t} = \lim_{t\to\infty} \frac{\sum_{n=1}^{N(t)} X_n^2}{N(t)} \frac{N(t)}{2t} = \frac{\mathsf{E}\left[X^2\right]}{2\mathsf{E}\left[X\right]} \quad \textbf{WP1}$$

Why is this true? It is an abbreviation for a sample-path result.

For the sample point ω, if the limits exist, we have

$$\lim_{t\to\infty} \frac{\sum_{n=1}^{N(t,\omega)} X_n^2(\omega)}{2t} = \lim_{t\to\infty} \frac{\sum_{n=1}^{N(t,\omega)} X_n^2(\omega)}{N(t,\omega)} \frac{N(t,\omega)}{2t}$$

For the given ω and a given t, the RHS above is the product of 2 numbers, and as t increases, we are looking at the limit of a product of numerical functions of t.

For those ω in a set of probability 1, both those functions converge to finite values as $t \to \infty$. Thus the limit of the product is the product of the limits.

This is a good example of why the strong law, dealing with sample paths, is so powerful.

Residual life and duration are examples of renewal reward functions.

In general $\mathcal{R}(Z(t), X(t))$ specifies reward as a function of location in the local renewal interval.

Thus reward over a renewal interval is

$$R_n = \int_{S_{n-1}}^{S_n} \mathcal{R}(\tau - S_{n-1}, X_n)\, d\tau = \int_{z=0}^{X_n} \mathcal{R}(z, X_n)\, dz$$

$$\lim_{t\to\infty} \frac{1}{t} \int_{\tau=0}^{t} R(\tau)\, d\tau = \frac{\mathsf{E}[R_n]}{\overline{X}} \qquad \textbf{W.P.1}$$

This also works for ensemble averages.

Def: A stopping trial (or stopping time) J for a sequence $\{X_n;\, n \geq 1\}$ of rv's is a positive integer-valued rv such that for each $n \geq 1$, the indicator rv $\mathbb{I}_{\{J=n\}}$ is a function of $\{X_1, X_2, \dots, X_n\}$.

A possibly defective stopping trial is the same except that J might be a defective rv. For many applications of stopping trials, it is not initially obvious whether J is defective.

Theorem (Wald's equality) Let $\{X_n; n \geq 1\}$ be a sequence of IID rv's, each of mean $\overline{X}$. If J is a stopping trial for $\{X_n;\, n \geq 1\}$ and if $\mathsf{E}[J] < \infty$, then the sum $S_J = X_1 + X_2 + \cdots + X_J$ at the stopping trial J satisfies

$$\mathsf{E}[S_J] = \overline{X}\mathsf{E}[J].$$

Wald: Let $\{X_n; n \geq 1\}$ be IID rv's, each of mean $\overline{X}$. If J is a stopping time for $\{X_n; n \geq 1\}$, $\mathsf{E}[J] < \infty$, and $S_J = X_1 + X_2 + \cdots + X_J$, then

$$\mathsf{E}[S_J] = \overline{X}\mathsf{E}[J]$$

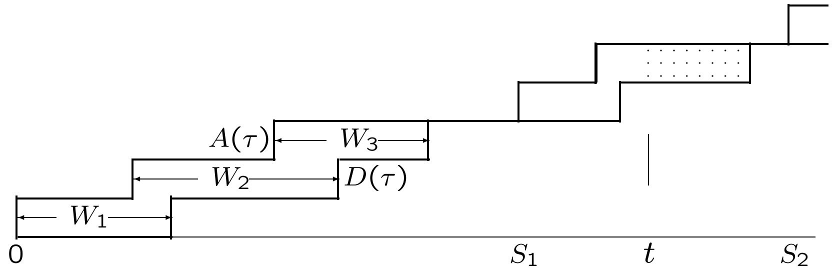
In many applications, where X_n and S_n are nonnegative rv's , the restriction $\mathsf{E}[J] < \infty$ is not necessary.

For cases where X is positive or negative, it is necessary as shown by 'stop when you're ahead.'

Little's theorem

This is an accounting trick plus some intricate handling of limits. Consider an queueing system with arrivals and departures where renewals occur on arrivals to an empty system.

Consider $L(t) = A(t) - D(t)$ as a renewal reward function. Then $L_n = \sum W_i$ over each busy period.



Let $\overline{L}$ be the time average number in system,

$$\overline{L} = \lim_{t\to\infty} \frac{1}{t}\int_0^t L(\tau)\,d\tau = \lim_{t\to\infty} \frac{\sum_{i=0}^{N(t)} W_i}{t}$$

$$\lambda = \lim_{t\to\infty} \frac{1}{t} A(t)$$

$$\begin{aligned} \overline{W} &= \lim_{t\to\infty} \frac{1}{A(t)} \sum_{i=1}^{A(t)} W_i \\ &= \lim_{t\to\infty} \frac{t}{A(t)} \lim_{t\to\infty} \frac{1}{t} \sum_{i=1}^{A(t)} W_i \\ &= \overline{L}/\lambda \end{aligned}$$

This is the same use of sample path limits as before.

Markov chains and renewal processes

For any finite-state ergodic Markov chain $\{X_n; n \geq 0\}$ with $X_0 = i$, there is a renewal counting process $\{N_i(t);\, t \geq 1\}$ where $N_i(t)$ is the number of visits to state i from time 1 to t. Let $Y_1, Y_2, \ldots$ be the inter-renewal periods. By the elementary renewal thm,

$$\lim_{t\to\infty} \frac{\mathsf{E}[N_i(t)]}{t} = \frac{1}{\overline{\overline{Y}}}$$

$$P_{ii}^t = \Pr\{N_i(t) - N_i(t-1) = 1\} = \mathsf{E}[N_i(t) - N_i(t-1)]$$

$$\sum_{n=1}^{t} P_{ii}^n = \mathsf{E}[N_i(t)]$$

But since $P_{ii}^t \to \pi_i$ exponentially,

$$\pi_i = \lim_{t\to\infty} \frac{\sum_{n=1}^{t} P_{ii}^t}{t} = \frac{\mathsf{E}[N_i(t)]}{t} = \frac{1}{\overline{\overline{Y}}}$$

Thus the mean recurrence time of state i is $1/\pi_i$.

Expected number of renewals, $m(t) = \mathsf{E}[N(t)]$

The elementary renewal theorem says

$$\lim_{t\to\infty} \mathsf{E}[N(t)]/t = 1/\overline{X}$$

For finite t, $m(t)$ can be very messy. Suppose the interarrival interval X is 1 or $\sqrt{2}$. As t increases, the points at which t can increase get increasingly dense, and $m(t)$ is non-decreasing but otherwise ugly.

Some progress can be made by expressing $m(t)$ in terms of its values at smaller t by the 'renewal equation.'

$$\begin{aligned} m(t) &= \mathsf{F}_X(t) + \int_0^t m(t-x)\,d\mathsf{F}_X(x); \qquad m(0) = 0 \\ &= \int_0^t [1 + m(t-x)]\mathsf{f}_X(x)\,dx \qquad \textbf{if } \mathsf{f}_X(x) \textbf{ exists} \end{aligned}$$

The renewal equation is linear in the function $m(t)$ and looks like equations in linear systems courses. It can be solved if $\mathsf{f}_X(x)$ has a rational Laplace transform. The solution has the form

$$m(t) = \frac{t}{\overline{X}} + \frac{\sigma^2}{2\overline{X}^2} - \frac{1}{2} + \epsilon(t) \qquad \textbf{for } t \geq 0,$$

where $\lim_{t\to\infty} \epsilon(t) = 0$.

The most significant term for large t is $t/\overline{X}$, consistent with the elementary renewal thm. The next two terms say the initial transient never quite dies away.

Heavy tailed distribution pick up extra renewals initially (recall $\mathsf{p}_X(\epsilon) = 1-\epsilon$, $\mathsf{p}_X(1/\epsilon) = \epsilon$).

Blackwell's theorem

Blackwell's theorem essentially says that the expected renewal rate for large t is $1/\overline{X}$.

It cannot quite say this, since if X is discrete, then S_n is discrete for all n. Thus suggests that $m(t) = \mathsf{E}[N(t)]$ does not have a derivative.

Fundamentally, there are two kinds of distribution funtions — arithmetic and non-arithmetic.

A rv X has an arithmetic distribution if its set of possible sample values are integer multiples of some number, say λ. The largest such choice of λ is the span of the distribution.

If X is arithmetic with span $\lambda > 0$, then every S_n must be arithmetic with a span either λ or an integer multiple of λ.

Thus $N(t)$ can increase only at multiples of λ.

For a non-arithmetic discrete distribution (example: $\mathsf{f}_X(1)=1/2$, $\mathsf{f}_X(\pi)=1/2$), the points at which $N(t)$ can increase become dense as $t \to \infty$.

Blackwell's thm:

$$\lim_{t\to\infty} [m(t+\lambda) - m(t)] = \frac{\lambda}{\overline{X}} \qquad \textbf{Arith. } X\textbf{, span } \lambda$$

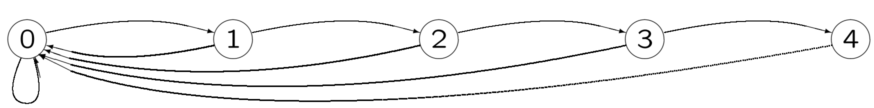
$$\lim_{t\to\infty} [m(t+\delta) - m(t)] = \frac{\delta}{\overline{X}} \qquad \textbf{Non-Arith. } X\textbf{, any } \delta > 0$$

Blackwell's theorem uses difficult analysis and doesn't lead to much insight. If Laplace techniques work, then it follows from the solution there.

The hard case is non-arithmetic but discrete distributions.

The arithmetic case with a finite set of values is easy. We model the renewal process as returns to a given state in a Markov chain. Choose $\lambda = 1$ for simplicity.

For any renewal process with inter-renewals at a finite set of integers times, there is a corresponding Markov chain modeling returns to state 0.



The transition probabilities can be seen to be

$$P_{i,i+1} = \frac{1 - \mathsf{p}_X(0) - \mathsf{p}_X(1) - \cdots - \mathsf{p}_X(i)}{1 - \mathsf{p}_X(0) - \mathsf{p}_X(1) - \cdots - \mathsf{p}_X(i-1)}$$

Assuming that the chain is aperiodic, we know that $\lim_{n\to\infty} P_{00}^n = \pi_0$. As seen before, $\pi_0 = 1/\overline{X}$.

Moral of story: When doing renewals, think Markov, and when doing Markov, think renewals.

Delayed renewal processes

A delayed renewal process is a modification of a renewal process for which the first inter-renewal interval X_1 has a different distribution than the others. They are still all independent.

The bottom line here is that all the limit theorems remain unchanged, even if $\mathsf{E}[X_1] = \infty$.

When modelling returns to a given state for a Markov chain, this lets us start in one state and count visits to another state.