MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Electrical Engineering and Computer Science

6.262 Discrete Stochastic Processes
Midterm Quiz
April 6, 2010

There are 5 questions, each with several parts. If any part of any question is unclear to you, ***please ask.***

The blue books are for scratch paper only. Don't hand them in. Put your final answers in the white booklets and briefly explain your reasoning for every question. **Please put your name on each white booklet you turn.**

Few questions require extensive calculations and most require very little, provided you pick the right tool or model in the beginning. The best approach to each problem is to first think carefully over what you've learned and decide precisely what tool fits best - before putting pencil to paper.

Partial Credit

We will give partial credit if you present your thinking in a clear way we can understand (and your thinking is at least partially correct), but otherwise not. If you model a problem using a tool that requires significant computation, it is best to first give the model explicitly and indicate how you will use the results of the computation to determine the final answer. This approach will help you receive fair credit if your computations aren't perfect.

Time

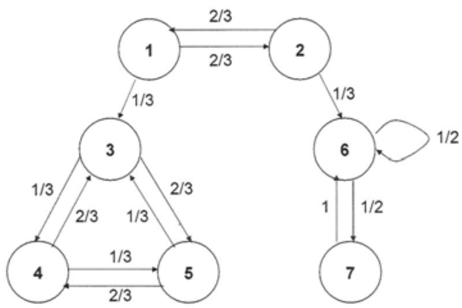
You will have at least 4 hours to finish the exam, and 30 – 60 minutes more if many of you request a bit more time.

Useful Tables and Formulas

These are given on the last pages of the quiz.

Problem 1 (19 pts.)

Consider the following finite-state Markov Chain.



(5) a) Identify all the classes present in the chain and the states belonging to each class. Find the period of each class and determine whether the class is transient or recurrent.

b) Let $p_{i,j}(n)$ denote the probability of the process ending up in state *j* in *n* transitions, conditioned on the fact that the process started in state *i*. In other words, $p_{i,j}(n) = P(X_n = j | X_0 = i)$. Compute the value of each of the limits below, or else explain briefly why it does not exist.

(2) i) $\lim_{n\to\infty} p_{1,5}(n)$.

(2) ii) $\lim_{n\to\infty} p_{1,7}(n)$.

(2) iii) $\lim_{n\to\infty} p_{1,2}(n)$.

(2) iv) $\lim_{n\to\infty} p_{4,5}(n)$.

(6) c) Let $P = [p_{i,j}]$ be the transition matrix for this chain. Find ***all*** the possible steady state vectors for this chain, i.e., find ***all*** vectors $\pi = [\pi_1, \pi_2, \ldots, \pi_7]$ with the properties that $\pi_1 + \pi_2 + \ldots + \pi_7 = 1,\ 0 \le \pi_1, \ldots, \pi_7 \le 1$ and $\pi P = \pi$.

Problem 2 (20 pts)

Consider a car ferry that holds some integer number *k* of cars and carries them across a river. The ferry business has been good, but customers complain about the long wait for the ferry to fill up. The cars arrive according to a renewal process. The ferry departs immediately upon the arrival of the *k*-th customer and subsequent ferries leave immediately upon the arrival of the 2*k*-th customer, the 3*k*-th customer, etc.

(5) a) The IID inter-arrival times of the cars have mean $\bar{X}$, variance σ^2 and moment generating function $g_X(r)$. Does the sequence of departure times of the ferries form a renewal process? Explain carefully.

(5) b) Find the expected time that a randomly chosen customer waits from arriving at the ferry terminal until departure of its ferry. As part of your solution, please give a reasonable definition of the expected waiting time for a randomly chosen customer, and please first solve this problem explicitly for the cases k = 1 and k = 2.

(5) c) Is there a 'slow truck' phenomenon here ? (This the phrase we used to describe the effect of large variance on the term $E[X^2]/2E[X]$ in the steady state residual life or on the $E[Z^2]$ term in the numerator of the Pollazcek-Khinchin formula.) Give a brief intuitive explanation.

(5) d) In an effort to decrease waiting, the ferry managers institute a policy where the maximum interval between ferry departures is 1 hour. Thus a ferry leaves either when it is full or after one hour has elapsed, whichever comes first. Does the sequence of departure times of ferries that leave with a full load of cars constitute a renewal process? Explain carefully.

Problem 3 (22pts)

The use of the various laws of large numbers with random variables that take huge values with tiny probabilities requires careful thought.

Except where exact answers are requested, your numerical answers need only be accurate to within $\pm$ 1%.)

Consider a discrete r.v. *X* with the PMF

$$p_X(-1) = (1 - 10^{-10})/2,$$
$$p_X(1) = (1 - 10^{-10})/2,$$
$$p_X(10^{12}) = 10^{-10}.$$

(4) a) Find the mean and variance of *X*. Assuming that $\{X_m;\ m \geq 1\}$ is an IID sequence with the distribution of *X* and that $S_n = X_1 + \cdots + X_n$ for each *n*, find the mean and variance of S_n.

(4) b) Sketch the distribution function of S_n for $n = 10^6$. (You may plot it as if it were a continuous function, but use a linear scale for the x-axis.) Estimate the value of *s* to within $\pm$ 1% for which $F_{S_n}(s) = 3/4$ and draw your sketch from -2s to 2s on the horizontal axis.

(4) c) Again for $n = 10^6$, find an exact expression for $1 - F_{S_n}(2 \text{ x } 10^6)$ when n = 10^6 and give a simple numerical approximation of this value (a better approximation than 0).

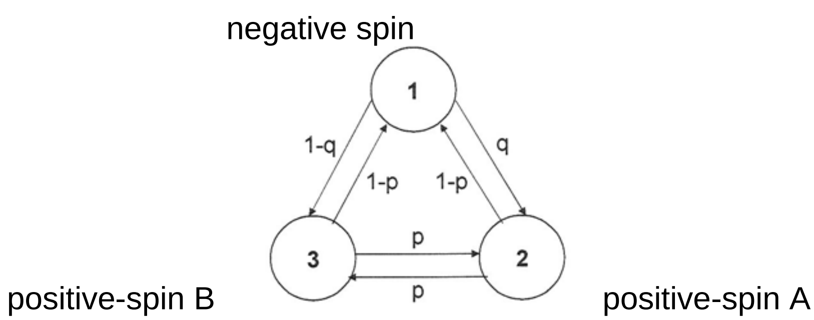
(5) d) Now let $n = 10^{10}$. Give an exact expression for $P(S_n \leq 10^{10})$ for n = 10^{10}, and find an approximate numerical value. Sketch the distribution function of S_n for $n = 10^{10}$. Scale the horizontal axis to include the points 0 near its left end and 2 x 10^{12} near its right end.

(5) e**)** What is roughly (i.e., within an order of magnitude or so) the smallest value of n for which the central limit theorem would work well for this problem ?

<u>Problem 4</u> (20 pts)

A certain physical object emits three types of particles: negative-spin particles, positive-spin particles of type A and positive-spin particles of type B. A processing device records the times and the types of particle arrivals at the sensor.

Considering only the particle types (i.e. negative, positive A, positive B), the corresponding arrival process can be modeled as a finite-state Markov chain shown below, where State 1 is associated with a negative-spin particle, State 2 with a positive-spin particle of type A and State 3 with a positive-spin particle of type B. For example, after recording a negative-spin particle, the next arrival at the sensor will be a positive-spin type A arrival with probability *q* or a positive-spin type B arrival with probability 1-*q*. Suppose that $p \in (0,1/2)$, $q \in (0,1/2)$..



a) Let us consider several possible processes arising from the above setup.

(4) i) Do the arrivals in time of positive-spin particles of type A form a renewal process? A delayed renewal process?

(4) ii) Do the arrivals in time of positive-spin particles form a renewal process?
A delayed renewal process?

(4) iii) Does the discrete-time process that records the spin orientations (i.e positive vs. negative) of the incoming particles form a Bernoulli process?

(4) iv) Does the discrete-time process that records the types (A vs. B) of incoming *positive-spin* particles form a Bernoulli proces

(4) b) Starting from State 1, let M_1 denote the number of transitions until the chain returns to State 1. Find $E(M_1)$ as well as the probability mass function of M_1.

Problem 5 (19 pts)

Consider a first-come, first serve M/M/1 queue with a customer arrival rate λ and a service rate μ, i.e.,

$$P(T_k > \tau) = e^{-\mu\tau}, \ \tau \geq 0,$$

where T_k is the service time for the k-th customer. Assume the queue is empty at t = 0, i.e., no customer is in the queue or in service at t = 0.

(1) a) Find the expected total wait in queue plus service for the first customer. (No derivation or explanation required.)

(5) b) Find the expected total wait in queue plus service for the second customer. (Please give complete calculation and briefly explain your reasoning.)

For the customer arrival process with rate λ, consider the age Z(t) of the interarrival interval at any time t ≥ 0. The first interarrival interval starts at t = 0. (This remaining three questions are about ***any*** Poisson process with rate λ, starting at t = 0, and have nothing to do with the queue above.)

(1) c) Find the expected age E[Z(0)] and find (or recall) $\lim_{t\to\infty} E[Z(t)]$. (No explanation required.)

(6) d) Find the expected age E[Z(t)], $\forall \ t \geq 0$. (You can derive this from your answer to part e), or you can solve part d) separately. Please give a complete calculation and briefly explain your reasoning.)

(6) e) Find the c $F_{Z(t)}(z)$, $\forall \ t \geq 0, \forall \ z \in [0,t]$. (Please give a complete calculation and briefly explain your reasoning.)

Possibly Useful Formula

For a large integer N and small ε, (i.e., N >> 1 and 0 < ε << 1),

$$(1 - \varepsilon)^N \approx e^{-N\varepsilon}$$

which also approximately equals, if Nε << 1 as well,

$$1 - N\varepsilon + \frac{N(N-1)}{2}\varepsilon^2 + ---$$

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