6.262 Discrete Stochastic Processes May 6, 2011, 2011
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Solutions to practice problem set 12

Note: There is a minor error in the statement of Exercise 7.21, part b. The last equation of that part should be $Z_J = -(-2)^n(2n-1)/(n^2-n)$. The error is corrected in the statement here.

Exercise 7.6 Consider a binary hypothesis testing problem where H is 0 or 1 and a one dimensional observation Y is given by $Y = H + U$ where U is uniformly distributed over [-1, 1] and is independent of H.

a) Find $\mathsf{f}_{Y|H}(y \mid 0)$, $\mathsf{f}_{Y|H}(y \mid 1)$ and the likelihood ratio $\Lambda(y)$.

Solution: Note that $f_{Y|H}$ is simply the density of U shifted by H, *i.e.*,

$$\mathsf{f}_{Y|H}(y \mid 0) \;=\; \begin{cases} 1/2; & -1 \le y \le 1 \\ 0; & \text{elsewhere} \end{cases} \qquad \mathsf{f}_{Y|H}(y \mid 1) \;=\; \begin{cases} 1/2; & 0 \le y \le 2 \\ 0; & \text{elsewhere} \end{cases} .$$

The likelihood ratio $\Lambda(y)$ is defined only for $-1 \le y \le 2$ since neither conditional density is nonzero outside this range.

$$\Lambda(y) = \frac{\mathsf{f}_{Y|H}(y \mid 0)}{\mathsf{f}_{Y|H}(y \mid 1)} \;=\; \begin{cases} \infty; & -1 \le y < 0 \\ 1; & 0 \le y \le 1 \\ 0; & 1 < y \le 2 \end{cases}$$

b) Find the threshold test at η for each η, $0 < \eta < \infty$ and evaluate the conditional error probabilities, $q_0(\eta)$ and $q_1(\eta)$.

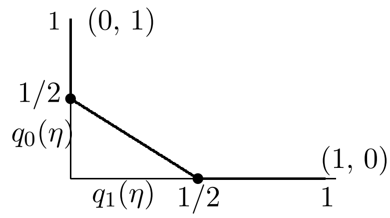
Solution: Since $\Lambda(y)$ has finitely many (3) possible values, all values of η between any adjacent pair lead to the same threshold test. Thus, for $\eta > 1$, $\Lambda(y) > \eta$, leads to the decision $\hat{h} = 0$ if and only if (iff) $\Lambda(y) = \infty$, *i.e.*, iff $-1 \le y < 0$. For $\eta = 1$, the rule is the same, $\Lambda(y) > \eta$ iff $\Lambda(y) = \infty$, but here there is a 'don't care' case $\Lambda(y) = 1$ where $0 \le y \le 1$ leads to $\hat{h} = 1$ simply because of the convention for the equal case taken in (7.14). Finally for $\eta < 1$, $\Lambda(Y) > \eta$ iff $-1 \le y \le 1$.

Consider $q_0(\eta)$ (the error probability conditional on $H = 0$ when a threshold η is used) for $\eta > 1$. Then $\hat{h} = 0$ iff $-1 \le y < 0$, and thus an error occurs (for $H = 0$) iff $y \ge 0$. Thus $q_0(\eta) = \Pr\{Y \ge 0 \mid H = 0\} = 1/2$. An error occurs given $H = 1$ (still assuming $\eta > 1$) iff $-1 \le y < 0$. These values of y are impossible under $H = 1$ so $q_1(\eta) = 0$. These error probabilities are the same if $\eta = 1$ because of the handling of the don't care cases.

For $\eta < 1$, $\hat{h} = 0$ iff $y \le 1$. Thus $q_0(\eta) = \Pr\{Y > 1 \mid H = 0\} = 0$. Also $q_1(\eta) = \Pr\{Y \le 1 \mid H = 1\} = 1/2$.

c) Find the error curve $u(\alpha)$ and explain carefully how $u(0)$ and $u(1/2)$ are found (hint: $u(0) = 1/2$).

Solution: We have seen that each $\eta \geq 1$ maps into the pair of error probabilities $(q_0(\eta), q_1(\eta)) = (1/2,\, 0)$. Similarly, each $\eta < 1$ maps into the pair of error probabilities $(q_0(\eta), q_1(\eta)) = (0,\, 1/2)$. The error curve contains these points and also contains the straight lines joining these points as shown below (see Figure 7.5). The point $u(\alpha)$ is the value of $q_0(\eta)$ for which $q_1(\eta) = \alpha$. Since $q_1(\eta) = 0$ for $\eta \geq 1$, $q_0(\eta) = 1/2$ for those values of η and thus $u(0) = 1/2$. Similarly, $u(1/2) = 0$.



d) Describe a decision rule for which the error probability under each hypothesis is 1/4. You need not use a randomized rule, but you need to handle the don't-care cases under the threshold test carefully.

Solution: The don't care cases arise for $0 \leq y \leq 1$ when $\eta = 1$. With the decision rule of (7.14), these don't care cases result in $\hat{h} = 1$. If half of those don't care cases are decided as $\hat{h} = 0$, then the error probability given $H = 1$ is increased to 1/4 and that for $H = 0$ is decreased to 1/4. This could be done by random choice, or just as easily, by mapping $y > 1/2$ into $\hat{h} = 1$ and $y \leq 1/2$ into $\hat{h} = 0$.

Exercise 7.12 a) Use Wald's equality to show that if $\overline{X} = 0$, then $\mathsf{E}[S_J] = 0$ where J is the time of threshold crossing with one threshold at $\alpha > 0$ and another at $\beta < 0$.

Solution: Wald's equality holds since $\mathsf{E}[|J|] < \infty$, which follows from Lemma 7.5.1. Thus $\mathsf{E}[S_J] = \bar{X}\mathsf{E}[J]$. Since $\bar{X} = 0$, it follows that $\mathsf{E}[S_J] = 0$.

b) Obtain an expression for $\Pr\{S_J \geq \alpha\}$. Your expression should involve the expected value of S_J conditional on crossing the individual thresholds (you need not try to calculate these expected values).

Solution: Writing out $\mathsf{E}[S_J] = 0$ in terms of conditional expectations,

$$\begin{aligned} \mathsf{E}[S_J] &= \Pr\{S_J \geq \alpha\}\,\mathsf{E}[S_J \mid S_J \geq \alpha] + \Pr\{S_J \leq \beta\}\,\mathsf{E}[S_J \mid S_J \leq \beta] \\ &= \Pr\{S_J \geq \alpha\}\,\mathsf{E}[S_J \mid S_J \geq \alpha] + [1 - \Pr\{S_J \geq \alpha\}]\mathsf{E}[S_J \mid S_J \leq \beta] \end{aligned}$$

Using $\mathsf{E}[S_J] = 0$, we can solve this for $\Pr\{S_J \geq \alpha\}$,

$$\Pr\{S_J \geq \alpha\} = \frac{\mathsf{E}[-S_J \mid S_J \leq \beta]}{\mathsf{E}[-S_J \mid S_J \leq \beta] + \mathsf{E}[S_J \mid S_J \geq \alpha]}$$

c) Evaluate your expression for the case of a simple random walk.

Solution: A simple random walk moves up or down only by unit steps, Thus if α and β are integers, there can be no overshoot when a threshold is crossed. Thus $\mathsf{E}[S_J \mid S_J \geq \alpha] = \alpha$ and $\mathsf{E}[S_J \mid S_J \leq \beta] = \beta$. Thus $\Pr\{S_J \geq \alpha\} = \frac{|\beta|}{|\beta|+\alpha}$. If α is non-integer, then a positive threshold crossing occurs at $\lceil\alpha\rceil$ and a lower threshold crossing at $\lfloor\beta\rfloor$. Thus, in this general case $\Pr\{S_J \geq \alpha\} = \frac{|\lfloor\beta\rfloor|}{|\lfloor\beta\rfloor|+\lceil\alpha\rceil}$.

d) Evaluate your expression when X has an exponential density, $\mathsf{f}_X(x) = a_1 e^{-\lambda x}$ for $x \geq 0$ and $\mathsf{f}_X(x) = a_2 e^{\mu x}$ for $x < 0$ and where a_1 and a_2 are chosen so that $\overline{X} = 0$.

Solution: Let us condition on $J = n, S_n \geq \alpha$, and $S_{n-1} = s$, for $s < \alpha$. The overshoot, $V = S_J - \alpha$ is then $V = X_n + s - \alpha$. Because of the memoryless property of the exponential, the density of V, conditioned as above, is exponential and $\mathsf{f}_V(v) = \lambda e^{-\lambda v}$ for $v \geq 0$. This does not depend on n or s, and is thus the overshoot density conditioned only on $S_J \geq \alpha$. Thus $\mathsf{E}[S_J \mid J \geq \alpha] = \alpha + 1/\lambda$. In the same way, $\mathsf{E}[S_J \mid S_J \leq \beta] = \beta - 1/\mu$. Thus

$$\Pr\{S_J \geq \alpha\} = \frac{|\beta| + \mu^{-1}}{\alpha + \lambda^{-1} + |\beta| + \mu^{-1}}$$

Note that it is not necessary to calculate a_1 or a_2.

Exercise 7.17 Suppose $\{Z_n;\, n \geq 1\}$ is a martingale. Show that

$$\mathsf{E}\left[Z_m \mid Z_{n_i}, Z_{n_{i-1}}, \ldots, Z_{n_1}\right] = Z_{n_i} \quad \text{for all } 0 < n_1 < n_2 < \ldots < n_i < m.$$

Solution: First observe from Lemma 7.6.1 that

$$\mathsf{E}[Z_m \mid Z_{n_i}, Z_{n_i-1}, Z_{n_i-2}, Z_1] = Z_{n_i}$$

This is valid for every sample value of every conditioning variable. Thus consider Z_{n_i-1} for example. Since this equation has the same value for each sample value of Z_{n_i-1}, we could take the expected value of this conditional expectation over Z_{n_i-1}, getting $\mathsf{E}[Z_m \mid Z_{n_i}, Z_{n_i-2}, Z_1] = Z_{n_i}$. In the same way, any subset of these conditioning rv's could be removed, leaving us with the desired form.

Exercise 7.21: a) This exercise shows why the condition $\mathsf{E}[|Z_J|] < \infty$ is required in Lemma 7.8.1. Let $Z_1 = -2$ and, for $n \geq 1$, let $Z_{n+1} = Z_n[1 + X_n(3n+1)/(n+1)]$ where $X_1, X_2, \ldots$ are IID and take on the values $+1$ and -1 with probability $1/2$ each. Show that $\{Z_n;\, n \geq 1\}$ is a martingale.

Solution: From the definition of Z_n above,

$$\mathsf{E}[Z_n \mid Z_{n-1}, Z_{n-2}, \ldots, Z_1] = \mathsf{E}[Z_{n-1}[1 + X_{n-1}(3n-2)/n] \mid Z_{n-1}, \ldots, Z_1]$$

Since the X_n are zero mean and IID, this is just $\mathsf{E}[Z_{n-1} \mid Z_{n-1} \ldots, Z_1]$, which is Z_{n-1}. Thus $\{Z_n;\, n \geq 1\}$ is a martingale.

b) Consider the stopping trial J such that J is the smallest value of $n > 1$ for which Z_n and Z_{n-1} have the same sign. Show that, conditional on $n < J$, $Z_n = (-2)^n/n$ and, conditional on $n = J$, $Z_J = -(-2)^n(n-2)/(n^2 - n)$.

Solution: It can be seen from the iterative definition of Z_n that Z_n and Z_{n-1} have different signs if $X_{n-1} = -1$ and have the same sign if $X_{n-1} = 1$. Thus the stopping trial is the smallest trial $n \geq 2$ for which $X_{n-1} = 1$. Thus for $n < J$, we must have $X_i = -1$ for $1 \leq i < n$. For $n = 2 < J$, X_1 must be -1, so from the formula above, $Z_2 = Z_1[1 - 4/2] = 2$. Thus $Z_n = (-2)^n/n$ for $n = 2 < J$. We can use induction now for arbtrary $n < J$. Thus for $X_n = -1$,

$$Z_{n+1} \;=\; Z_n \left[1 - \frac{3n+1}{n+1}\right] \;=\; \frac{(-2)^n}{n} \cdot \frac{-2n}{n+1} \;=\; \frac{(-2)^{n+1}}{n+1}$$

The remaining task is to compute Z_n for $n = J$. Using the result just derived for $n = J-1$ and using $X_{J-1} = 1$,

$$Z_J \;=\; Z_{J-1} \left[1 + \frac{3(J-1)+1}{J}\right] \;=\; \frac{(-2)^{J-1}}{J-1} \cdot \frac{4J-2}{J} \;=\; \frac{-(-2)^J(2J-1)}{J(J-1)}$$

c) Show that $\mathsf{E}[|Z_J|]$ is infinite, so that $\mathsf{E}[Z_J]$ does not exist according to the definition of expectation, and show that $\lim_{n\to\infty} \mathsf{E}[Z_n|J > n] \Pr\{J > n\} = 0$.

Solution: We have seen that $J = n$ if and only if $X_i = -1$ for $1 \leq i \leq n-2$ and $X_{n-1} = 1$. Thus $\Pr\{J = n\} = 2^{-n+1}$ so

$$\mathsf{E}[|Z_J|] \;=\; \sum_{n=2}^{\infty} 2^{n-1} \cdot \frac{2^n(2n-1)}{n(n-1)} \;=\; \sum_{n=2}^{\infty} \frac{2(2n-1)}{n(n-1)} \;\geq\; \sum_{n=2}^{\infty} \frac{4}{n} \;=\; \infty,$$

since the harmonic series diverges.

Finally, we see that $\Pr\{J > n\} = 2^{n-1}$ since this event occurs if and only if $X_i = -1$ for $1 \leq i < n$. Thus

$$\mathsf{E}[Z_n \mid J > n] \Pr\{J > n\} \;=\; \frac{2^{-n+1}2^n}{n} \;=\; 2/n \;\to 0$$

Section 7.8 explains the significance of this exercise.

Exercise 7.29 Let $\{Z_n;\, n \geq 1\}$ be a martingale, and for some integer m, let $Y_n = Z_{n+m} - Z_m$.

a) Show that $\mathsf{E}[Y_n \mid Z_{n+m-1} = z_{n+m-1}, Z_{n+m-2} = z_{n+m-2}, \ldots, Z_m = z_m, \ldots, Z_1 = z_1] = z_{n+m-1} - z_m$.

Solution: This is more straightforward if the desired result is written in the more abbreviated form

$$\mathsf{E}[Y_n \mid Z_{n+m-1}, Z_{n+m-2}, \ldots, Z_m, \ldots, Z_1] = Z_{n+m-1} - Z_m.$$

Since $Y_n = Z_{n+m} - Z_m$, the left side above is

$$\mathsf{E}[Z_{n+m} - Z_m | Z_{n+m-1}, \ldots, Z_1] = Z_{n+m-1} - \mathsf{E}[Z_m \mid Z_{n+m-1}, \ldots, Z_m, \ldots, Z_1]$$

Given sample values for each conditioning rv on the right of the above expression, and particularly given that $Z_m = z_m$, the expected value of Z_m is simply the conditioning value z_m for Z_m. This is one of those strange things that are completely obvious, and yet somehow obscure. We then have $\mathsf{E}[Y_n \mid Z_{n+m-1}, \dots, Z_1] = Z_{n+m-1} - Z_m$.

b) Show that $\mathsf{E}[Y_n \mid Y_{n-1} = y_{n-1}, \dots, Y_1 = y_1] = y_{n-1}$.

Solution: In abbreviated form, we want to show that $\mathsf{E}[Y_n \mid Y_{n-1}, \dots, Y_1] = Y_{n-1}$. We showed in part a) that $\mathsf{E}[Y_n \mid Z_{n+m-1}, \dots, Z_1] = Y_{n-1}$. For each sample point ω, $Y_{n-1}(\omega), \dots, Y_1(\omega)$ is a function of $Z_{n+m-1}(\omega), \dots, Z_1(\omega)$. Thus, the rv $\mathsf{E}[Y_n \mid Z_{n+m-1}, \dots, Z_1]$ specifies the rv $\mathsf{E}[Y_n \mid Y_{n-1}, \dots, Y_1]$, which then must be Y_{n-1}.

c) Show that $\mathsf{E}[|Y_n|] < \infty$. Note that **b)** and **c)** show that $\{Y_n;\, n \geq 1\}$ is a martingale.

Solution: Since $Y_n = Z_{n+m} - Z_m$, we have $|Y_n| \leq |Z_{n+m}| + |Z_m|$. Since $\{Z_n;\, n \geq 1$ is a martingale, $\mathsf{E}[|Z_n|] < \infty$ for each n so

$$\mathsf{E}[|Y_n|] \leq \mathsf{E}[|Z_{n+m}|] + \mathsf{E}[|Z_m|] < \infty.$$

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