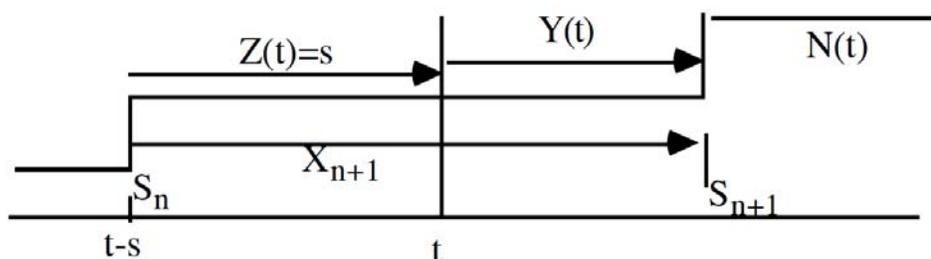# Solutions to Homework 8
6.262 Discrete Stochastic Processes
MIT, Spring 2011

Exercise 4.11

a) From the figure, conditional on $S_n = t - s$ (i.e., conditional on the age at time t being
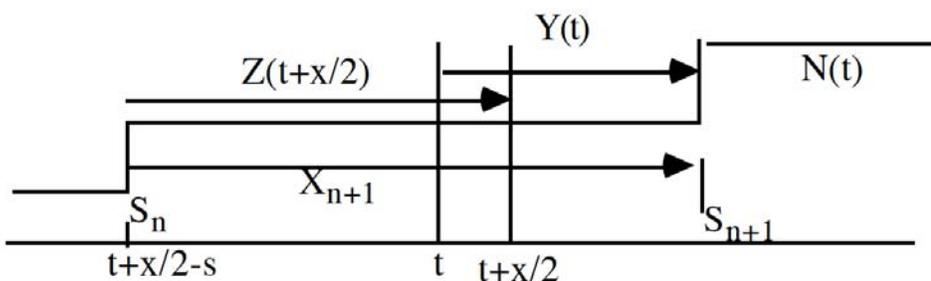

s and on $N(t) = n$), the probability that the next arrival occurs after time $t + x$ is

$$\Pr\{Y(t) > x | Z(t) = s, N(t) = n\} = \frac{\Pr\{X_{n+1} > s + x\}}{\Pr\{X_{n+1} > s\}} = \frac{1 - F_X(s + x)}{1 - F_X(s)}, s \geq 0$$

Since this does not depend on n, the condition $N(t) = n$ can be omitted, and this is also $\Pr\{Y(t) > x | Z(t) = s\}$.

b)

If $s < x/2$, then there was an arrival in $[t, t + x/2]$ and hence $Y(t)$ is less than x, giving us $\Pr\{Y(t) > x | Z(t + x/2) = s\} = 0$.

If $s \geq x/2$ and if, to be specific, we take $N(t) = n$, then the condition $\{Z(t + \frac{x}{2}) = s, N(t) = n\}$ means that both $S_n = t - s + \frac{x}{2}$ and $S_{n+1} > t + \frac{x}{2}$, i.e.,
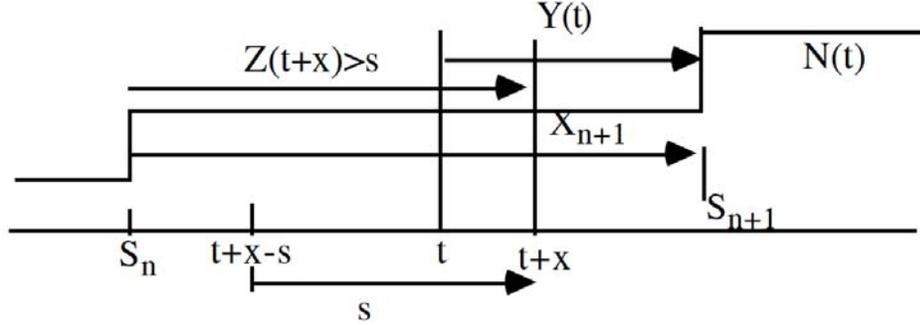
$$\{Z(t + \frac{x}{2}) = s, N(t) = n\} = \{S_n = t - s + \frac{x}{2}, X_{n+1} > s\}$$

The condition $Y(t) > x$ can be translated under these conditions to $X_{n+1} > x + s$. Thus,

1

$$\Pr\{Y(t) > x | Z(t+x/2) = s, N(t) = n\} = \frac{\Pr\{X_{n+1} > s + x/2\}}{\Pr\{X_{n+1} > s\}} = \frac{1 - F_X(s + x/2)}{1 - F_X(s)}, s \geq 0$$

Again, this does not depend on n, so it is also $\Pr\{Y(t) > x | Z(t+x/2) = s\}$.

c)

From the figure, if $s > x$, the event $\{Z(t+x) > s\}$ means that there are no arrivals in the interval $(t+x-s, t+x)$, which implies no arrivals from t to $t+x$, so that $Y(t)$ must exceed x. Thus, $\Pr\{Y(t) > x | Z(t+x) > s\} = 1$. If $s < x$, the event $\{Z(t+x) > s\}$ means that there were no arrivals in $[t+x-s, t+x]$. Thus, the probability that $Y(t) \geq x$ is the probability that there are no arrivals in the interval $[t, t+x-s]$. Since $X(t)$ is a Poisson process,

$$\Pr\{Y(t) > x | Z(t+x) > s\} = \exp(-\lambda(x-s)), \text{ for } s \leq x.$$

Exercise 5.1

We use induction on n. As the basis for the induction, we know that $F_{ij}(1) = P_{ij}$. Since the x_i's by assumption non-negative, it follows for all i that

$$F_{ij}(1) = P_{ij} \leq P_{ij} + \sum_{k \neq j} P_{kj} x_j = x_i$$

Now assume that for a given $n \geq 1$, $F_{ij}(n) \leq x_i$ for all i. Using (5.9),

$$\begin{aligned} F_{ij}(n+1) &= P_{ij} + \sum_{k \neq j} P_{kj} F_{kj}(n) \\ &\leq P_{ij} + \sum_{k \neq j} P_{kj} x_j = x_i, \text{ for all } i. \end{aligned}$$

From (5.7), $F_{ij}(n)$ is non-decreasing in n and thus has a limit, $F_{ij}(\infty) \le x_i$ for all i.

Exercise 5.2

a) It may be helpful before verifying these equations to explain where they come from. This derivation essentially solves the problem also, but the verification to follow, using the hint, is valuable to verify the solution, especially for those just becoming familiar with this topic.

First let $F_{10}(\infty)$, the probability of ever reaching state 0 starting from state 1, be denoted as α. Since α is determined solely by transitions from states $i \ge 1$, and since each state i "sees" the same Markov chain for states $j \ge i$, we know that $F_{i,i-1}(\infty) = \alpha$ for all $i \ge 1$. Thus, using a trick we have used before, $\alpha = q + p\alpha^2$. That is, the probability of ever reaching 0 from 1 is q (the probability of going there immediately) plus p (the probability of going to state 2) times α (the probability of ever getting back to 1 from state 2) times α (the probability of then getting from 1 to 0).

This quadratic has two solutions, $\alpha = 1$ and $\alpha = q/p$. With the help of Exercise 5.1, we know that $F_{10}(\infty) = q/p$, i.e., the smaller of the two solutions. From this solution, we immediately deterine that $F_{00}(\infty) = q + p\alpha = 2q$. Also, for each $i > 1$, $F_{i0}(\infty)$ is the probability of ever moving from i to $i - 1$ times that of ever moving from $i - 1$ to $i - 2$, etc. down to 0. Thus $F_{i0}(\infty) = \alpha^i$. We next verify this solution algebraically.

We write the set of equations (5.10) for $j = 0$, as $x_i = P_{i0} + \sum_{k \ne 0} P_{ik}x_k$; $i \ge 0$. For the chain at hand this set of equations simplifies to

$$
\begin{aligned}
x_0 &= q + px_1 \\
x_1 &= q + px_2 \\
x_i &= qx_{i-1} + px_{i+1}; \text{ for } i \ge 2
\end{aligned}
$$

(1)

First we show that $X_0 = 2q$ and $x_i = (q/p)^i$ satisfies these equations:

$$
\begin{aligned}
i = 0 \quad &: \quad q + p(q/p) = 2q = x_0 \\
i = 1 \quad &: \quad q + p(q/p)^2 = q/p = x_1 \\
i \ge 2 \quad &: \quad q(q/p)^{i-1} + p(q/p)^{i+1} = (q/p)^i = x_i
\end{aligned}
$$

From exercise 5.1, the true $F_{i0}(\infty)$ satisfies $F_{i0}(\infty) \le x_i$ for any solution $\{x_i\}$ of ?????. For $p = 1$, the solution above is 0 for all i, so it is clearly the smallest non-negative solution. For $1/2 \le p < 1$, we now explore whether there are other solutions for $\{x_i\}$ such that $x_0 \le 2q$, and $x_i \le (q/p)^i$ for all $i \ge 1$. The equations (1) can be written

$$
\begin{aligned}
x_1 &= (x_0 - q)/p; \\
x_2 &= (x_1 - q)/p; \\
x_{i+1} &= (x_i - qx_{i-1})/p; \text{ for } i \ge 2
\end{aligned}
$$

(2)

This shows that all x_i are specified iteratively in terms of x_0. Thus, if there is a smaller solution than the one above, it must have $x_0 = 2q - \epsilon$. Defining δ_i by $x_i = (q/p)^i - \delta_i$ in the solution of (2) with $x_0 = 2q - \epsilon$, we get $\delta_1 = \epsilon/p$ and $\delta_2 = \epsilon/p^2$. We next use induction to show that $\delta_{i+1} > \delta_i$ for all $i \geq 1$. $\delta_2 > \delta_1$ is established above, so we assume $\delta_i > \delta_{i-1}$ for given $i \geq 2$. Then,

$$
\begin{aligned}
x_{i+1} &= \left\{ \left[(q/p)^i - \delta_i \right] - q \left[(q/p)^{i-1} - \delta_{i-1} \right] \right\} / p < \left\{ \left[(q/p)^i - \delta_i \right] - q \left[(q/p)^{i-1} - \delta_i \right] \right\} / p \\
&= (q/p)^{i+1} - \delta_i \left[1/p - q/p \right] = (q/p)^{i+1} - \delta_i
\end{aligned}
$$

Since $x_{i+1} = (q/p)^{i+1} - \delta_{i+1} < (q/p)^{i+1} - \delta_i$, we have $\delta_{i+1} > \delta_i$. Thus, $\delta_i > \epsilon$ for all i, and since $(q/p)^i$ approaches 0 with increasing i, $x_{i+1} < 0$ for sufficiently large i. Thus, there is no solution to (1) with $x_0 < 2q$ and the solution above is the true set $\{F_{i0}(\infty)\}$.

b) Starting in any given state $j > 0$, the first transition goes to $j - 1$ with probability q, and to $j + 1$ with probability p. If this first transition is to $j - 1$, then there is an eventual return to state j (consider the original chain truncated at state j). On the other hand, if this first transition is to state $j + 1$, then the situation until the first return to j (if such a return occurs) is the same as if the states 1 to $j - 1$ were eliminated. Thus, $F_{jj}(\infty) = F_{00}(\infty) = 2q$. Similarly, for $i > 0$, $F_{j+i,j}(\infty) = F_{i,0}(\infty) = (q/p)^i$.

Exercise 5.3:

a) This is the Chapman-Kolmogorov equality in the form $P_{ij}^n = \sum_k P_{ik}^{n-1} P_{kj}$ where $P_{kj} = p$ for $k = j - 1$, $P_{kj} = q$ for $k = j + 1$ and $P_{kj} = 0$ elsewhere.

b) This is less tedious if organized in an array of terms. Each term (except P_{00}^n) for each n is then half the term to the upper left plus half the term to the upper right. P_{00}^n is half the term above plus half the term to the upper right. P_{00}^n is half the term above plus half the term to the upper right.

j	0	1	3	3	4
$P_{0,j}^1$	1/2	1/2			
$P_{0,j}^2$	1/2	1/4	1/4		
$P_{0,j}^3$	3/8	3/8	1/8	1/8	
$P_{0,j}^4$	3/8	1/4	1/4	1/16	1/16

From (5.3), for $n = 4$, $j = 0$, $j + n$ is even, so $P_{00}^4 = \binom{4}{2} 2^{-4} = 6/16$, which agrees with the table above. For $j = 1$, $j + n$ is odd, so according to (5.3), $P_{01}^4 = \binom{4}{3} 2^{-4} = 4/16$, which again agrees with the table. The other terms are similar. Using induction to validate (5.3)

in general, we assume that (5.3) is valid for a given n and all j. First, for $j = 0$ and n even, we have

$$P_{00}^{n+1} = \frac{1}{2}[P_{00}^n + P_{01}^n] = \frac{1}{2}\left[\binom{n}{n/2}2^{-n} + \binom{n}{(n/2)+1}2^{-n}\right]$$

For arbitrary positive integer $n > k$, a useful combinatorial identity is

$$\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}$$

This can be seen by writing out all the factorials and manipulating the equations, but can be seen more simply by viewing $\binom{n+1}{k+1}$ as the number of ways to arrange $k+1$ ones in a binary $n+1$ tuple. These arrangements can be separated into those that start with a one followed by k ones out of n and those that start with a zero followed by $k+1$ ones out of n.

Applying this identity, we have $P_{00}^{n+1} = \binom{n+1}{(n/2)+1}2^{-(n+1)}$. Since $n+1+j$ is odd with $j = 0$, this agrees with (5.3). Next, for n even and $j \geq 2$ even, we have

$$
\begin{aligned}
P_{0j}^{n+1} &= \frac{1}{2}\left[P_{0,j-1}^n + P_{0,j+1}^n\right] = \frac{1}{2}\left[\binom{n}{(j+n)/2}2^{-n} + \binom{n}{((j+n)/2)+1}2^{-n}\right] \\
&= \binom{n+1}{((j+n)/2)+1}2^{-(n+1)}
\end{aligned}
$$

c) The symmetry to be used here will be more clear if the states are labelled as $\cdots, -3/2, -1/2, 1/2, 3/2, \cdots$ for the Bernoulli, type chain and $1/2, 3/2, 5/2, \cdots$ for the M/M/1 type chain.
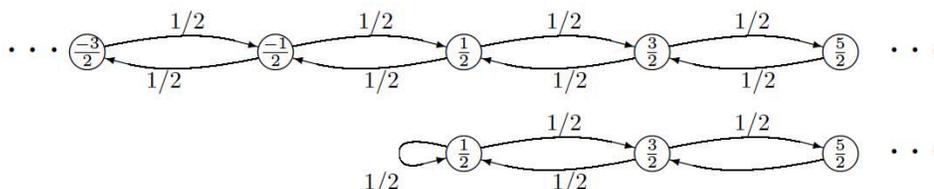
Now let $\{X_n; n \geq 0\}$ be the sequence of states for the Bernoulli type chain above and let $Y_n = |X_n|$ for each n. Since all the transition probabilities are $1/2$ above, it can be seen that $\{Y_n; n \geq 0\}$ is a Markov chain, and is in fact the M/M/1 type chain. That is, except for state $1/2$, there is an equiprobable move up by 1 or down by 1. If $X_n = 1/2$, then X_{n+1} is $3/2$ or $-1/2$ with equal probability, so $|X_n|$ is $1/2$ or $3/2$ with equal probability.

This means that to find $P_{1/2,j+1/2}^n$ for the M/M/1 type chain with $p = 1/2$, we find $P_{1/2,j+1/2}^n$ and $P_{1/2,-j-1/2}^n$ for the Bernoulli chain and add them together.

There is one final peculiarity here: the Bernoulli chain is periodic, but each positive state $j + 1/2$ is in the opposite subclass from $-j - 1/2$, so that only one of these two terms is nonzero for each n.

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