Solutions to Homework 7

6.262 Discrete Stochastic Processes

MIT, Spring 2011

Exercise 4.10: Consider a variation of an M/G/1 queueing system in which there is no facility to save waiting customers. Assume customers arrive according to a Poisson process of rate λ. If the server is busy, the customer departs and is lost forever; if the server is not busy, the customer enters service with a service time distribution function denoted by $\mathsf{F}_Y(y)$.

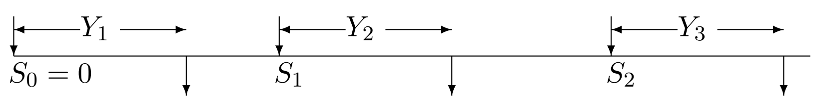
Successive service times (for those customers that are served) are IID and independent of arrival times. Assume that customer number 0 arrives and enters service at time $t = 0$.

a) Show that the sequence of times $S_1, S_2, \dots$ at which successive customers enter service are the renewal times of a renewal process. Show that each inter-renewal interval $X_i = S_i - S_{i-1}$ (where $S_0 = 0$) is the sum of two independent random variables, $Y_i + U_i$ where Y_i is the ith service time; find the probability density of U_i.

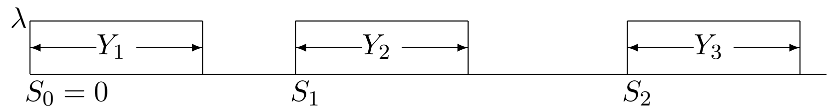
Solution: Let Y_1 be the first service time, *i.e.*, the time spent serving customer 0. Customers who arrive during $(0, Y_1]$ are lost, and, given that $Y_1 = y$, the residual time until the next customer arrives is memoryless and exponential with rate λ. Thus the time $X_1 = S_1$ at which the next customer enters service is $Y_1 + U_1$ where U_1 is exponential with rate λ, *i.e.*, $\mathsf{f}_{U_1}(u) = \lambda \exp(-\lambda u)$.

At time X_1, the arriving customer enters service, customers are dropped until $X_1 + Y_2$, and after an exponential interval U_2 of rate λ a new customer enters service at time $X_1 + X_2$ where $X_2 = Y_2 + U_2$. Both Y_2 and U_2 are independent of X_1, so X_2 and X_1 are independent. Since the Y_i are IID and the U_i are IID, X_1 and X_2 are IID. In the same way, the sequence $X_1, X_2, \dots,$ are IID intervals between successive services. Thus $\{X_i;\, i \geq 1\}$ is a sequence of inter-renewal intervals for a renewal process and $S_1, S_2, \dots,$ are the renewal epochs.

b) Assume that a reward (actually a cost in this case) of one unit is incurred for each customer turned away. Sketch the expected reward function as a function of time for the sample function of inter-renewal intervals and service intervals shown below; the expectation is to be taken over those (unshown) arrivals of customers that must be turned away.



Solution: Customers are turned away at rate λ during the service times, so that $R(t)$, the reward (the reward at time t averaged over dropped customer arrivals but for a given sample path of services and residual times) is given by $R(t) = \lambda$ for t in a service interval and $R(t) = 0$ otherwise.



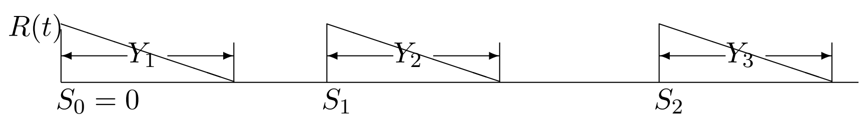
Note that the number of arrivals within a service interval are dependent on the length of the service interval but independent of arrivals outside of that interval and independent of other service intervals.

c) Let $\int_0^t R(\tau)d\tau$ denote the accumulated reward (i.e., cost) from 0 to t and find the limit as $t \to \infty$ of $(1/t)\int_0^t R(\tau)d\tau$. Explain (without any attempt to be rigorous or formal) why this limit exists with probability 1.

Solution: The reward within the nth inter-renewal interval (as a random variable over that interval) is $R_n = \lambda Y_n$. Using Theorem 4.4.1, then, the sample average reward, WP1, is $\frac{\lambda \mathsf{E}[Y]}{\mathsf{E}[Y]+1/\lambda}$.

d) In the limit of large t, find the expected reward from time t until the next renewal. Hint: Sketch this expected reward as a function of t for a given sample of inter-renewal intervals and service intervals; then find the time-average.

Solution: For the sample function above, the reward to the next inter-renewal (again averaged over droppped arrivals) is given by

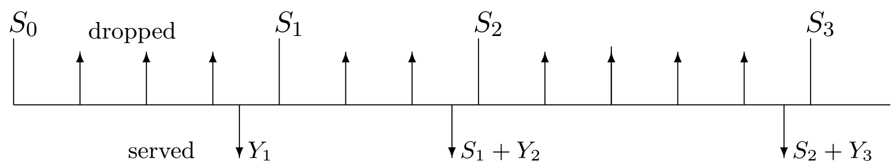


The reward over the nth inter-renewal interval is then $\lambda S_n^2/2$ so the sample path average of the expected reward per unit time is

$$\frac{\mathsf{E}[R(t)]}{\overline{X}} = \frac{\lambda \mathsf{E}\left[Y^2\right]}{2(\overline{Y}+1/\lambda)}$$

e) Now assume that the arrivals are deterministic, with the first arrival at time 0 and the nth arrival at time $n-1$. Does the sequence of times $S_1, S_2, \ldots$ at which subsequent customers start service still constitute the renewal times of a renewal process? Draw a sketch of arrivals, departures, and service time intervals. Again find $\lim_{t\to\infty}\left(\int_0^t R(\tau)\,d\tau\right)/t$.

Solution: Since the arrivals are deterministic at unit intervals, the packet to be served at the completion of Y_1 is the packet arriving at $\lceil Y_1 \rceil$ (the problem statement was not sufficiently precise to specify what happends if a service completion and a customer arrival are simultaneous, so we assume here that such a customer is served). The customers arriving from time 1 to $\lceil Y_1 \rceil - 1$ are then dropped as illustrated below.



Finding the sample path average as before,

$$\frac{\mathsf{E}\left[R(t)\right]}{\overline{X}} = \frac{\mathsf{E}\left[\lceil Y \rceil - 1\right]}{\mathsf{E}\left[\lceil Y \rceil\right]}$$

Exercise 4.12:

In order to find the sample-average distribution function, $F_Z(z)$ of the age of a renewal process, we can define a reward function similar to the one in example 4.4.5 as follows:

$$R(t) = \mathcal{R}(Z(t), \tilde{X}(t)) = \begin{cases} 1; & Z(t) \le z \\ 0; & \text{otherwise} \end{cases}$$

We see that $R(t)$ has positive value only over the first z unit of interval, thus $R_n = \min(z, X_n)$. Hence,

$$\mathbb{E}[R_n] = \mathbb{E}[\min(z, X_n)] = \int_{x=0}^{\infty} \Pr\{\min(X, z) > x\}\ \mathrm{d}x = \int_{x=0}^{z} \Pr\{X > x\}\ \mathrm{d}x$$

Now we know that $F_Z(z) = \lim_{t\to\infty} 1/t \int_0^t R(\tau)\,\mathrm{d}\tau$ is the time-average fraction of time that the age is less than or equal to z. From theorem 4.4.1, we get:

$$F_Z(z) = \frac{\mathbb{E}[R_n]}{\overline{X}} = \frac{1}{\overline{X}} \int_{x=0}^{z} \Pr\{X > x\}\ \mathrm{d}x \text{ , WP1}$$

Exercise 4.14:

a) We define the sequence of random variables X_i corresponding to each outcome of the Bernouli experiment with probability of success of p. So, X_i's are IID and $X_i = 1$ with probability p and $X_i = 0$ with probability $1 - p$. Thus, $\mathbb{E}[X] = p$.

$S_n = X_1 + X_2 + \cdots + X_n$ is the number of successes of Bernouli process in first n experiments.

We define J the stopping time as the number of trials up to and including the kth success. (Can you prove that it is actually a valid stopping time?). So we know that $S_J = k$ since the process is stopped when there has been k successes in the process. Hence, $\mathbb{E}[S_J] = k$.

Wald equality says that $\mathbb{E}[S_J] = \mathbb{E}[X]\mathbb{E}[J]$. Thus, the expected number of trials up to and including the kth success is

$$\mathbb{E}[J] = \frac{\mathbb{E}[S_J]}{\mathbb{E}[X]} = \frac{k}{p}$$

b) First, we calculate the expected number of trials up to and including the first success in the Bernouli process (We call this random variable T).

$$\begin{aligned} \mathbb{E}[T] &= \mathbb{E}[T|X_1 = 1]\Pr\{()\, X_1 = 1) + \mathbb{E}[T|X_1 = 0]\Pr\{()\, X_1 = 0) \\ &= 1 \times p + (1 + \mathbb{E}[T])(1 - p) \end{aligned}$$

$$\mathbb{E}[T] = \frac{1}{p}$$

Since the trials of this experiment are independent, the expected time up to and including the kth success is k times the expected time up to and including first success. This is the exact same result as part (a) which says $\mathbb{E}[J] = k/p$

<u>Exercise 4.15:</u>

a) The weak law is sufficient. J is the smallest value of n for which $S_n = X_1 + X_2 + \cdots + X_n \le -d$. Thus, $\Pr\{J > n\} = \Pr\{S_1 > -d, S_2 > -d, \cdots, S_n > -d\} \le \Pr\{S_n > -d\}$. And we know that $\Pr\{S_n > -d\} = \Pr\left\{S_n/n - \overline{X} > -\overline{X} - d/n\right\}$. The weak law, on the other hand, says that $\lim_{n\to\infty} \Pr\left\{|S_n/n - \overline{X}| \ge \epsilon\right\} = 0$ for any $\epsilon > 0$. Taking $\epsilon = -\overline{X}/2$, and restricting attention to $n \ge d/\epsilon$, we have:

$$\Pr\left\{S_n/n - \overline{X} > -\overline{X} - d/n\right\} \le \Pr\left\{S_n/n - \overline{X} \ge \epsilon\right\} \le \Pr\left\{|S_n/n - \overline{X}| \ge \epsilon\right\} \to 0$$

b) One stops playing on trial $J = n$ if one's capital reaches 0 for the first time on the nth trial, *i.e.*, if $S_n = -d$ for the first time at trial n. This is clearly a function of $X_1, X_2, \ldots, X_n$, so J is a stopping rule. Note that stopping occurs exactly on reaching $-d$ since S_n can decrease with n only in increments of -1 and S_n is always integer. Thus $S_J = -d$

Using Wald's equality, which is valid since $\mathbb{E}[J] < \infty$, we have

$$\mathbb{E}[J] = \frac{-d}{\overline{X}}$$

which is positive since $\overline{X}$ is negative. You should note from the exercises we have done with Wald's equality that it is usually used to solve for $E[J]$ after determining $E[S_J]$.

Exercise 4.17:

Let $J = \min\{n \mid S_n \leq b \text{ or } S_n \geq a\}$, where a is a positive integer, b is a negative integer, and $S_n = X_1 + X_2 + \cdots + X_n$. Assume that $\{X_i;\, i \geq 1\}$ is a set of zero mean IID rv's that can take on only the set of values $\{-1, 0, +1\}$, each with positive probability.

a) Is J a stopping rule? Why or why not? Hint: The more difficult part of this is to argue that J is a random variable (*i.e.*, non-defective); you do not need to construct a proof of this, but try to argue why it must be true.

Solution: For J to be a stopping trial, it must be a random variable and also $\mathbb{I}_{J=n}$ must be a function of $X_1, \ldots, X_n$. Now S_n is a function of $X_1, \ldots, X_n$, so the event that $S_n \geq a$ or $S_n \leq b$ is a function of S_n and the first n at which this occurs is a function of $S_1, \ldots, S_n$. Thus $\mathbb{I}_{J=n}$ must be a function of $X_1, \ldots, X_n$. For J to be a rv, we must show that $\lim_{n\to\infty} \Pr\{J \leq n\} = 1$ The central limit theorem states that $(S_n - n\overline{X})/\sqrt{n}\sigma$ approaches a normalized Gaussian rv in distribution as $n \to \infty$. Since $\overline{X} = 0$, $S_n/\sqrt{n}\sigma$ must approach normal. Now both $a/\sqrt{n}\sigma$ and $b/\sqrt{n}\sigma$ approach 0, so the probability that S_n (*i.e.*, the process without stopping) is between these limits goes to 0 as $n \to \infty$. Thus the probability that the process has not stopped by time n goes to 0 as $n \to \infty$.

An alternate approach here is to model $\{S_n; n \geq 1\}$ for the stopped process as a Markov chain where a and b are recurrent states and the other states are transient. Then we know that one of the recurrent states are reached eventually with probability 1.

b) What are the possible values of S_J?

Solution: Since S_n can change only in integer steps, it cannot exceed a without first equaling a and it cannot be less than b without first equaling b. Thus S_J is only a or b.

c) Find an expression for $\mathsf{E}[S_J]$ in terms of p, a, and b, where $p = \Pr\{S_J \geq a\}$.

Solution: $\mathsf{E}[S_J] = a\Pr\{S_J = a\} + b\Pr\{S_J = b\} = pa + (1-p)b$

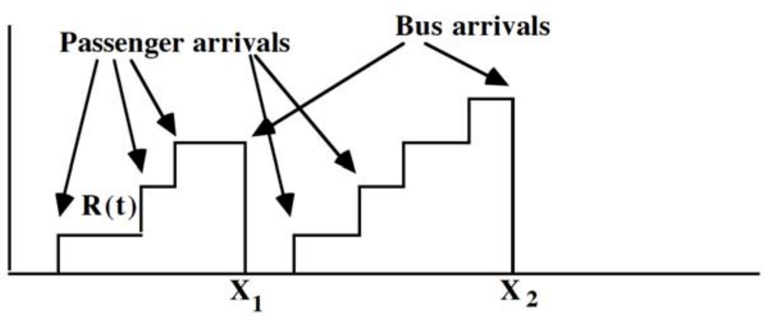
d) Find an expression for $\mathsf{E}[S_J]$ from Wald's equality. Use this to solve for p.

Solution: Since J is a stopping trial for $X_1, X_2, \ldots$ and the X_i are IID, we have $\mathsf{E}[S_J] = \overline{X}\mathsf{E}[J]$. Since $\overline{X} = 0$, we conclude that $\mathsf{E}[S_J] = 0$. Combining this with part c), we have $0 = pa + (1-p)b$, so $p = -b/(a-b)$. This is easier to interpret as $p = |b|/(a + |b|)$.

This approach works only for $\overline{X} = 0$, but we will see in Chapter 7 how to solve the problem for an arbitrary distribution on X. We also note that the solution is independent of the probability of the self loop. Finally we note that this helps explain the peculiar behavior of the 'stop when you're ahead' example. The negative threshold b represents the capital of the gambler in that example and shows that as $b \to -\infty$, the probability of reaching the threshold a increases, but at the expense of a larger catastrophe if the gamblers capital is wiped out.

Exercise 4.28

a)



b) Given that the first bus arrives at x, the number of customers picked up is the number of Poisson customer arrivals in $(0, x]$. The expected number of these arrivals is λx, so the expected number picked up is λx. Similarly, $\mathbb{E}[R(t)|\text{first bus at } x] = \lambda t$ for $0 < t < x$. Thus,

$$\mathbb{E}[\int_0^x R(t)\,\mathrm{d}t | X_1 = x] = \int_0^x \lambda t\,\mathrm{d}t = \lambda x^2/2$$

c) From theorem 4.4.1,

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t R(\tau)\,\mathrm{d}\tau = \frac{\mathbb{E}[R_n]}{\mathbb{E}[X]} = \frac{\lambda \mathbb{E}[X^2]}{2\mathbb{E}[X]}$$

Where we have used part (b), averaged over X_1, to see that $\mathbb{E}[R_n] = \mathbb{E}[R_1] = \lambda \mathbb{E}[X^2]/2$. This is the time average number of waiting customers. Assuming that F_X is non-arithmetic, this is $\lim_{t\to\infty} \mathbb{E}[R(t)]$, which is the ensemble average number of waiting customers at time t in the limit $t \to \infty$. (i.e., in steady state).

d) Consider a reward function with a reward for each customer equal to the wait of that customer. Given that a customer arrives at time t, the wait of that customer is the residual life $Y(t)$ of the bus arrival process. Averaging over the Poisson customer arrivals for a given sample function of the bus arrival process, the time average reward is $\lim_{t\to\infty} (\lambda/t) \int_0^t Y(\tau)\, d\tau$ which is $\lambda\mathbb{E}[X^2]/(2\mathbb{E}[X])$ with probability 1. Since the customer arrival rate λ, the time average wait per customer is $\mathbb{E}[X^2]/(2\mathbb{E}[X])$. An alternate approach is to observe that Little's relation holds here. The time average expected wait per customer is the time average number in the system divided by the arrival rate λ, which, using part (c), is $\mathbb{E}[X^2]/(2\mathbb{E}[X])$.

e) There are no customers at the bus stop from the time of a bus arrival until the next customer arrival. Consider a reward rate of 1 in each such period. Let X_n be the n-th, inter-renewal period and let U_n be the time from S_n (the start of the n-th renewal period) until the first subsequent customer arrival. The aggregate reward in the n-th renewal period is then $\min(X_n, U_n)$. Since X_n and U_n are independent, $\Pr\{\min(X_n, U_n) > t\} = \Pr\{X_n > t\} \Pr\{U_n > t\}$. We then have

$$\mathbb{E}[\min(X_n, U_n)] = \int_0^\infty \Pr\{\min(X_n, U_n) > t\}\; dt = \int_0^\infty \Pr\{X_n > t\}\, e^{-\lambda t}\, dt$$

Using theorem 4.4.1 again, the fraction of time there are no customers at the bus stop is $\frac{1}{\mathbb{E}[X_n]} \int_0^\infty \Pr\{X_n > t\}\, e^{-\lambda t} dt$. Note that if $\mathbb{E}[X_n] \ll 1/\lambda$, then $e^{-\lambda t} \approx 1$ for t such that $\Pr\{X_n > t\}$ is appreciable. Thus,

$$\int_0^\infty \Pr\{X_n > t\}\, e^{-\lambda t}\, dt \approx \int_0^\infty \Pr\{X_n > t\}\; dt = \mathbb{E}[X_n]$$

So the fraction of time with no customers is approximately 1, which makes sense. If the bus arrivals are Poisson with rate μ, the $\mathbb{E}[\min(X_n, U_n)] = 1/(\lambda + \mu)$, so the fraction of time with no customers is $\mu/(\lambda + \mu)$.

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